# Chapter II

# Measure and Integral

## 1 Classes of Sets

Given: a non-empty set $\Omega$ and a class $\mathfrak{A} \subset \mathfrak{P}(\Omega)$ of subsets. Put

$$\mathfrak{A}^+ = \Big\{\bigcup_{i=1}^n A_i : n \in \mathbb{N} \wedge A_1, \dots, A_n \in \mathfrak{A} \text{ pairwise disjoint}\Big\}.$$

**Definition 1.**

(i) $\mathfrak{A}$ *closed w.r.t. intersections* or $\cap$–*closed* iff $A, B \in \mathfrak{A} \Rightarrow A \cap B \in \mathfrak{A}$.

(ii) $\mathfrak{A}$ *closed w.r.t. unions* or $\cup$–*closed* iff $A, B \in \mathfrak{A} \Rightarrow A \cup B \in \mathfrak{A}$.

(iii) $\mathfrak{A}$ *closed w.r.t. complements* or ${}^c$–*closed* iff $A \in \mathfrak{A} \Rightarrow A^c := \Omega \setminus A \in \mathfrak{A}$.

(iv) $\mathfrak{A}$ *semi-algebra (in* $\Omega$*)* if

   (a) $\Omega \in \mathfrak{A}$,

   (b) $\mathfrak{A}$ $\cap$–closed,

   (c) $A \in \mathfrak{A} \Rightarrow A^c \in \mathfrak{A}^+$.

(v) $\mathfrak{A}$ *algebra (in* $\Omega$*)* if

   (a) $\Omega \in \mathfrak{A}$,

   (b) $\mathfrak{A}$ $\cap$–closed,

   (c) $\mathfrak{A}$ ${}^c$–closed.

(vi) $\mathfrak{A}$ $\sigma$*-algebra (in* $\Omega$*)* if

   (a) $\Omega \in \mathfrak{A}$,

   (b) $A_1, A_2, \ldots \in \mathfrak{A} \Rightarrow \bigcup_{n=1}^\infty A_n \in \mathfrak{A}$,

   (c) $\mathfrak{A}$ ${}^c$–closed.

**Remark 1.** Let $\mathfrak{A}$ denote a $\sigma$-algebra in $\Omega$. Recall that a *probability measure* $P$ on $(\Omega, \mathfrak{A})$ is a mapping

$$P : \mathfrak{A} \to [0, 1]$$

such that $P(\Omega) = 1$ and

$$A_1, A_2, \ldots \in \mathfrak{A} \text{ pairwise disjoint} \quad \Rightarrow \quad P\Big(\bigcup_{i=1}^{\infty} A_i\Big) = \sum_{i=1}^{\infty} P(A_i).$$

Moreover, $(\Omega, \mathfrak{A}, P)$ is called a *probability space*, and $P(A)$ is the *probability* of the *event* $A \in \mathfrak{A}$.

**Remark 2.**

(i) $\mathfrak{A}$ $\sigma$-algebra $\Rightarrow$ $\mathfrak{A}$ algebra $\Rightarrow$ $\mathfrak{A}$ semi-algebra.

(ii) $\mathfrak{A}$ closed w.r.t. intersections $\Rightarrow$ $\mathfrak{A}^+$ closed w.r.t. intersections.

(iii) $\mathfrak{A}$ algebra and $A_1, A_2 \in \mathfrak{A} \Rightarrow A_1 \cup A_2,\ A_1 \setminus A_2,\ A_1 \triangle A_2 \in \mathfrak{A}$.

(iv) $\mathfrak{A}$ $\sigma$-algebra and $A_1, A_2, \dots \in \mathfrak{A} \Rightarrow \bigcap_{n=1}^{\infty} A_n \in \mathfrak{A}$.

**Example 1.**

(i) Let $\Omega = \mathbb{R}$ and consider the class of intervals

$$\mathfrak{A} = \{]a, b] : a, b \in \mathbb{R} \wedge a < b\} \cup \{]-\infty, b] : b \in \mathbb{R}\} \cup \{]a, \infty[ : a \in \mathbb{R}\} \cup \{\mathbb{R}, \emptyset\}.$$

Then $\mathfrak{A}$ is a semi-algebra, but not an algebra.

(ii) $\{A \in \mathfrak{P}(\Omega) : A \text{ finite or } A^c \text{ finite}\}$ is an algebra, but not a $\sigma$-algebra in general.

(iii) $\{A \in \mathfrak{P}(\Omega) : A \text{ countable or } A^c \text{ countable}\}$ is a $\sigma$-algebra.

(iv) $\mathfrak{P}(\Omega)$ is the largest $\sigma$-algebra in $\Omega$, $\{\emptyset, \Omega\}$ is the smallest $\sigma$-algebra in $\Omega$.

**Definition 2.**

(i) $\mathfrak{A}$ *monotone class (in $\Omega$)* if

(a) $A_1, A_2, \ldots \in \mathfrak{A} \wedge A_n \uparrow A$[1] $\Rightarrow A \in \mathfrak{A}$,

(b) $A_1, A_2, \ldots \in \mathfrak{A} \wedge A_n \downarrow A$[2] $\Rightarrow A \in \mathfrak{A}$.

(ii) $\mathfrak{A}$ *Dynkin class (in $\Omega$)* if

(a) $\Omega \in \mathfrak{A}$,

(b) $A_1, A_2 \in \mathfrak{A} \wedge A_1 \subset A_2 \Rightarrow A_2 \setminus A_1 \in \mathfrak{A}$,

(c) $A_1, A_2, \ldots \in \mathfrak{A}$ *pairwise disjoint* $\Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathfrak{A}$.

**Remark 3.** $\mathfrak{A}$ $\sigma$-algebra $\Rightarrow$ $\mathfrak{A}$ monotone class and Dynkin class.

[1] I.e., $A_n \subseteq A_{n+1}$ for all $n$ and $A = \bigcup_n A_n$
[2] I.e., $A_{n+1} \subseteq A_n$ for all $n$ and $A = \bigcap_n A_n$

**Theorem 1.**

(i) For every algebra $\mathfrak{A}$

$$\mathfrak{A}\ \sigma\text{-algebra} \quad \Leftrightarrow \quad \mathfrak{A} \text{ monotone class.}$$

(ii) For every Dynkin class $\mathfrak{A}$

$$\mathfrak{A}\ \sigma\text{-algebra} \quad \Leftrightarrow \quad \mathfrak{A} \text{ closed w.r.t. intersections.}$$

*Proof.* Ad (i), '$\Leftarrow$': Let $A_1, A_2, \ldots \in \mathfrak{A}$ and put $B_m = \bigcup_{n=1}^m A_n$ and $B = \bigcup_{n=1}^\infty A_n$. Then $B_m \uparrow B$. Furthermore, $B_m \in \mathfrak{A}$ since $\mathfrak{A}$ is an algebra. Thus $B \in \mathfrak{A}$ since $\mathfrak{A}$ is a monotone class.

Ad (ii), '$\Leftarrow$': For $A \in \mathfrak{A}$ we have $A^c = \Omega \setminus A \in \mathfrak{A}$ since $\mathfrak{A}$ is a Dynkin class. For $A, B \in \mathfrak{A}$ we have

$$A \cup B = A \cup (B \setminus (A \cap B)) \in \mathfrak{A}$$

since $\mathfrak{A}$ is also closed w.r.t. intersections. Thus, for $A_1, A_2, \ldots \in \mathfrak{A}$ and $B_m$ as previously we get $B_m \in \mathfrak{A}$ and

$$\bigcup_{n=1}^\infty A_n = \bigcup_{m=1}^\infty (B_m \setminus B_{m-1}) \in \mathfrak{A},$$

where $B_0 = \emptyset$. □

**Remark 4.** Consider $\sigma$-algebras (algebras, monotone classes, Dynkin classes) $\mathfrak{A}_i$ for $i \in I \neq \emptyset$. Then $\bigcap_{i\in I} \mathfrak{A}_i$ is a $\sigma$-algebra (algebra, monotone class, Dynkin class), too.

Given: a class $\mathfrak{E} \subset \mathfrak{P}(\Omega)$.

**Definition 3.** The *$\sigma$-algebra generated by* $\mathfrak{E}$

$$\sigma(\mathfrak{E}) = \bigcap \{\mathfrak{A} : \mathfrak{A}\ \sigma\text{-algebra in } \Omega \wedge \mathfrak{E} \subset \mathfrak{A}\}.$$

Analogously, $\alpha(\mathfrak{E})$, $m(\mathfrak{E})$, $\delta(\mathfrak{E})$ the *algebra, monotone class, Dynkin class,* respectively, *generated by* $\mathfrak{E}$.

**Remark 5.** For $\gamma \in \{\sigma, \alpha, m, \delta\}$ and $\mathfrak{E}, \mathfrak{E}_1, \mathfrak{E}_2 \subset \mathfrak{P}(\Omega)$

(i) $\gamma(\mathfrak{E})$ is the smallest '$\gamma$-class' that contains $\mathfrak{E}$,

(ii) $\mathfrak{E}_1 \subset \mathfrak{E}_2 \Rightarrow \gamma(\mathfrak{E}_1) \subset \gamma(\mathfrak{E}_2)$,

(iii) $\gamma(\gamma(\mathfrak{E})) = \gamma(\mathfrak{E})$.

**Example 2.** Let $\Omega = \mathbb{N}$ and $\mathfrak{E} = \{\{n\} : n \in \mathbb{N}\}$. Then

$$\alpha(\mathfrak{E}) = \{A \in \mathfrak{P}(\Omega) : A \text{ finite or } A^c \text{ finite}\} =: \mathfrak{A}.$$

Proof: $\mathfrak{A}$ is an algebra, see Example 1, and $\mathfrak{E} \subset \mathfrak{A}$. Thus $\alpha(\mathfrak{E}) \subset \mathfrak{A}$. On the other hand, for every finite set $A \subset \Omega$ we have $A = \bigcup_{n\in A} \{n\} \in \alpha(\mathfrak{E})$, and for every set $A \subset \Omega$ with finite complement we have $A = (A^c)^c \in \alpha(\mathfrak{E})$. Thus $\mathfrak{A} \subset \alpha(\mathfrak{E})$.

Moreover,

$$\sigma(\mathfrak{E}) = \mathfrak{P}(\mathbb{N}), \qquad m(\mathfrak{E}) = \mathfrak{E}, \qquad \delta(\mathfrak{E}) = \mathfrak{P}(\mathbb{N}).$$

**Theorem 2.** [Monotone class theorem, set version]

(i) $\mathfrak{E}$ closed w.r.t. intersections $\Rightarrow \sigma(\mathfrak{E}) = \delta(\mathfrak{E})$.

(ii) $\mathfrak{E}$ algebra $\Rightarrow \sigma(\mathfrak{E}) = m(\mathfrak{E})$.

*Proof.* Ad (i): Remark 3 implies

$$\delta(\mathfrak{E}) \subset \sigma(\mathfrak{E}).$$

We claim that

$$\delta(\mathfrak{E}) \text{ is closed w.r.t. intersections.} \tag{1}$$

Then, by Theorem 1.(ii),

$$\sigma(\mathfrak{E}) \subset \delta(\mathfrak{E}).$$

Put

$$\mathfrak{C}_B = \{C \subset \Omega : C \cap B \in \delta(\mathfrak{E})\}, \qquad B \in \delta(\mathfrak{E}),$$

so that (1) is equivalent to

$$\forall\, B \in \delta(\mathfrak{E}) : \delta(\mathfrak{E}) \subset \mathfrak{C}_B. \tag{2}$$

It is straightforward to verify that

$$\forall\, B \in \delta(\mathfrak{E}) : \mathfrak{C}_B \text{ Dynkin class.} \tag{3}$$

Moreover, since $\mathfrak{E}$ is closed w.r.t. intersections,

$$\forall\, E \in \mathfrak{E} : \mathfrak{E} \subset \mathfrak{C}_E.$$

Therefore

$$\forall\, E \in \mathfrak{E} : \delta(\mathfrak{E}) \subset \mathfrak{C}_E,$$

i.e., for all $E \in \mathfrak{E}, B \in \delta(\mathfrak{E})$, $E \cap B \in \delta(\mathfrak{E})$; hence

$$\forall\, B \in \delta(\mathfrak{E}) : \mathfrak{E} \subset \mathfrak{C}_B.$$

Since $\mathfrak{C}_B$ is a Dynkin system, $\delta(B) \subset \mathfrak{C}_B$.
Ad (ii): Obviously, $m(\mathfrak{E}) \subset \sigma(\mathfrak{E})$. By Part (ii) of Theorem 1, it is enough to show that $m(\mathfrak{E})$ is an algebra. This amounts to the claim that

$$m(\mathfrak{E}) \text{ is } {}^c\text{–closed and} \cap \text{–closed .} \tag{4}$$

First, the class

$$\mathfrak{C} := \{A \in m(\mathfrak{E}) \;:\; A^c \in m(\mathfrak{E})\}$$

is monotone, contains $\mathfrak{E}$ by assumption, and thus equals $m(\mathfrak{E})$. Second, in complete analogy to Part (i), for $B \in m(\mathfrak{E})$ it follows that the set

$$\mathfrak{C}_B = \{C \subset \Omega : C \cap B \in m(\mathfrak{E})\}$$

is a monotone class containing $\mathfrak{E}$ and thus $m(\mathfrak{E})$, so that $m(\mathfrak{E})$ is indeed $\cap$–closed. □

**Lemma 1.** $\mathfrak{E}$ semi-algebra $\Rightarrow \alpha(\mathfrak{E}) = \mathfrak{E}^+$.

*Proof.* Clearly $\mathfrak{E} \subset \mathfrak{E}^+ \subset \alpha(\mathfrak{E})$. It remains to show that $\mathfrak{E}^+$ is an algebra. For

$$A = \bigcup_{i=1}^{n} A_i \in \mathfrak{E}^+, \qquad A_i \in \mathfrak{E} \text{ disjoint},$$

$$B = \bigcup_{i=1}^{n} B_i \in \mathfrak{E}^+, \qquad B_i \in \mathfrak{E} \text{ disjoint},$$

$$A \cap B = \bigcup_{\substack{i\le n\\ j\le m}} (A_i \cap B_j), \qquad (A_i \cap B_j) \in \mathfrak{E} \text{ disjoint}.$$

Hence $\mathfrak{E}^+$ is $\cap$–stable. For

$$A = \bigcup_{i=1}^{n} A_i \in \mathfrak{E}^+, \qquad A_i \in \mathfrak{E} \text{ disjoint},$$

with

$$A_i^c = \bigcup_{j\le n_i} B_j^i, \qquad B_j^i \in \mathfrak{E} \text{ disjoint},$$

we have

$$\begin{aligned} A^c &= \bigcap_{i\le n} \bigcup_{j\le n_i} B_j^i \\ &= \bigcup_{\substack{(j_1,\dots,j_n)\\ j_i\le n_i}} \Bigg( \underbrace{\bigcap_{i=1}^{n} B_{j_i}^i}_{\in\mathfrak{E} \text{ disjoint}} \Bigg). \end{aligned}$$

Hence $A^c \in \mathfrak{E}^+$, and $\mathfrak{E}^+$ is an algebra. $\square$

Put

$$\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\},$$

and equip this with the metric $d(x,y) := |\arctan(x) - \arctan(y)|$. Then $\overline{\mathbb{R}}$ is a complete, compact, separable, order complete metric space. For $a \in \mathfrak{R}$ set

$$(\pm\infty) + (\pm\infty) = a + (\pm\infty) = (\pm\infty) + a = \pm\infty, \qquad a/_{\pm\infty} = 0,$$

$$a \cdot (\pm\infty) = (\pm\infty) \cdot a = \begin{cases} \pm\infty & \text{if } a > 0 \\ 0 & \text{if } a = 0 \\ \mp\infty & \text{if } a < 0 \end{cases}$$

as well as $-\infty < a < \infty$.

Recall that $(\Omega, \mathfrak{G})$ is a *topological space* iff $\mathfrak{G} \subset \mathfrak{P}(\Omega)$ satisfies

(i) $\emptyset, \Omega \in \mathfrak{G}$,

(ii) $\mathfrak{G}$ is closed w.r.t. to intersections,

(iii) for every family $(G_i)_{i\in I}$ with $G_i \in \mathfrak{G}$ we have $\bigcup_{i\in I} G_i \in \mathfrak{G}$.

$\mathfrak{G}$ is the set of *open subsets* of $\Omega$, and the complements of open sets are the *closed subsets* of $\Omega$. $K \subset \Omega$ is *compact* iff for every family $(G_i)_{i\in I}$ with $G_i \in \mathfrak{G}$ and

$$K \subset \bigcup_{i\in I} G_i$$

there is a finite set $I_0 \subset I$ such that

$$K \subset \bigcup_{i\in I_0} G_i.$$

For $\Omega = \mathbb{R}^k$ and $\Omega = \overline{\mathbb{R}}^k$, we consider the natural (product) topologies $\mathfrak{G}_k$, $\overline{\mathfrak{G}}_k$.

**Definition 4.** For every topological space $(\Omega, \mathfrak{G})$

$$\mathfrak{B}(\Omega) = \sigma(\mathfrak{G})$$

is the *Borel-$\sigma$-algebra (in $\Omega$ w.r.t. $\mathfrak{G}$)*. We shorten

$$\mathfrak{B} = \mathfrak{B}(\mathbb{R}), \quad \overline{\mathfrak{B}} = \mathfrak{B}(\overline{\mathbb{R}}), \quad \mathfrak{B}_k = \mathfrak{B}(\mathbb{R}^k), \overline{\mathfrak{B}}_k = \mathfrak{B}(\overline{\mathbb{R}}^k),$$

**Remark 6.** We have

$$\begin{aligned}\mathfrak{B}_k &= \sigma(\{F \subset \mathbb{R}^k : F \text{ closed}\}) = \sigma(\{K \subset \mathbb{R}^k : K \text{ compact}\}) \\ &= \sigma(\{]-\infty, a] : a \in \mathbb{R}^k\}) = \sigma(\{]-\infty, a] : a \in \mathbb{Q}^k\})\end{aligned}$$

and

$$\overline{\mathfrak{B}} = \{B \subset \overline{\mathbb{R}} : B \cap \mathbb{R} \in \mathfrak{B}\}. \tag{5}$$

One can prove that $\#\mathfrak{B}_k = \#\mathbb{R}^k$, and thus

$$\mathfrak{B}_k \subsetneq \mathfrak{P}(\mathbb{R}^k)$$

see Billingsley (1979, Exercise 2.21).

**Definition 5.** For any $\sigma$-algebra $\mathfrak{A}$ in $\Omega$ and $\widetilde{\Omega} \subset \Omega$

$$\widetilde{\mathfrak{A}} = \{\widetilde{\Omega} \cap A : A \in \mathfrak{A}\}$$

is the *trace-$\sigma$-algebra of* $\mathfrak{A}$ *in* $\widetilde{\Omega}$, sometimes denoted by $\widetilde{\Omega} \cap \mathfrak{A}$.

**Remark 7.**

(i) $\widetilde{\mathfrak{A}}$ is a $\sigma$-algebra in $\widetilde{\Omega}$.

(ii) $\widetilde{\mathfrak{A}} \not\subset \mathfrak{A}$ in general, but if $\widetilde{\Omega} \in \mathfrak{A}$, then $\widetilde{\mathfrak{A}} = \{A \in \mathfrak{A} : A \subset \widetilde{\Omega}\}$.

(iii) $\mathfrak{A} = \sigma(\mathfrak{E}) \Rightarrow \widetilde{\mathfrak{A}} = \sigma(\{\widetilde{\Omega} \cap E : E \in \mathfrak{E}\})$.

(iv) $\mathfrak{B}_k = \mathbb{R}^k \cap \overline{\mathfrak{B}}_k$, see (5) for $k = 1$.

(v) $[a, b[ \cap \mathfrak{B}_k = \sigma(\{[a, c[ : a \le c \le b\})$, see (iii).

# 2 Measurable Mappings

**Definition 1.** $(\Omega, \mathfrak{A})$ is called *measurable space* iff $\Omega \neq \emptyset$ and $\mathfrak{A}$ is a $\sigma$-algebra in $\Omega$. Elements $A \in \mathfrak{A}$ are called $(\mathfrak{A}-)$*measurable sets*.

In the sequel, $(\Omega_i, \mathfrak{A}_i)$ are measurable spaces for $i = 1, 2, 3$.

**Remark 1.** Let $f : \Omega_1 \to \Omega_2$. For $B \in \mathfrak{A}_2$, we set in short

$$\{f \in B\} = f^{-1}(B) = \{\omega \in \Omega_1 \ : \ f(\omega) \in B\} \subset \Omega_1$$

(i) $f^{-1}(\mathfrak{A}_2) = \{f^{-1}(A) : A \in \mathfrak{A}_2\}$ is a $\sigma$-algebra in $\Omega_1$.

(ii) $\{A \subset \Omega_2 : f^{-1}(A) \in \mathfrak{A}_1\}$ is a $\sigma$-algebra in $\Omega_2$.

**Definition 2.** $f : \Omega_1 \to \Omega_2$ is $\mathfrak{A}_1$-$\mathfrak{A}_2$-*measurable* iff $f^{-1}(\mathfrak{A}_2) \subset \mathfrak{A}_1$. i.e., iff for all $A \in \mathfrak{A}_2$ we have $\{f \in A\} \in \mathfrak{A}_1$.

How can we prove measurability of a given mapping?

**Theorem 1.** If $f : \Omega_1 \to \Omega_2$ is $\mathfrak{A}_1$-$\mathfrak{A}_2$-measurable and $g : \Omega_2 \to \Omega_3$ is $\mathfrak{A}_2$-$\mathfrak{A}_3$-measurable, then $g \circ f : \Omega_1 \to \Omega_3$ is $\mathfrak{A}_1$-$\mathfrak{A}_3$-measurable.

*Proof.* (Compare Bemerkung 5.4,(i), Analysis IV)

$$(g \circ f)^{-1}(\mathfrak{A}_3) = f^{-1}(g^{-1}(\mathfrak{A}_3)) \subset f^{-1}(\mathfrak{A}_2) \subset \mathfrak{A}_1 \ .$$

□

**Lemma 1.** For $f : \Omega_1 \to \Omega_2$ and $\mathfrak{E} \subset \mathfrak{P}(\Omega_2)$

$$f^{-1}(\sigma(\mathfrak{E})) = \sigma(f^{-1}(\mathfrak{E})).$$

*Proof.* By $f^{-1}(\mathfrak{E}) \subset f^{-1}(\sigma(\mathfrak{E}))$ and Remark 1.(i) we get $\sigma(f^{-1}(\mathfrak{E})) \subset f^{-1}(\sigma(\mathfrak{E}))$.
Let $\mathfrak{F} = \{A \subset \Omega_2 : f^{-1}(A) \in \sigma(f^{-1}(\mathfrak{E}))\}$. Then $\mathfrak{E} \subset \mathfrak{F}$ and $\mathfrak{F}$ is a $\sigma$-algebra, see Remark 1.(ii). Thus we get $\sigma(\mathfrak{E}) \subset \mathfrak{F}$, i.e., $f^{-1}(\sigma(\mathfrak{E})) \subset \sigma(f^{-1}(\mathfrak{E}))$. □

**Theorem 2.** If $\mathfrak{A}_2 = \sigma(\mathfrak{E})$ with $\mathfrak{E} \subset \mathfrak{P}(\Omega_2)$, then

$$f \text{ is } \mathfrak{A}_1\text{-}\mathfrak{A}_2\text{-measurable} \quad \Leftrightarrow \quad f^{-1}(\mathfrak{E}) \subset \mathfrak{A}_1 \ .$$

*Proof.* (compare Lemma 5.2, Analysis IV) '⇒' is trivial,
'⇐':Assume that $f^{-1}(\mathfrak{E}) \subset \mathfrak{A}_1$. By Lemma 1,

$$f^{-1}(\mathfrak{A}_2) = f^{-1}(\sigma(\mathfrak{E})) = \sigma(f^{-1}(\mathfrak{E})) \subset \sigma(\mathfrak{A}_1) = \mathfrak{A}_1.$$

□

**Corollary 1.** Let $(\Omega_i, \mathfrak{G}_i)$ be topological spaces. Then every continuous $f : \Omega_1 \to \Omega_2$ is $\mathfrak{B}(\Omega_1)$-$\mathfrak{B}(\Omega_2)$-measurable.

*Proof.* (Compare Korollar 5.3, Analysis IV) For continuous $f$ we have

$$f^{-1}(\mathfrak{G}_2) \subset \mathfrak{G}_1 \subset \sigma(\mathfrak{G}_1) = \mathfrak{B}(\Omega_1).$$

Theorem 2 shows the claim. □

Given: measurable spaces $(\Omega_i, \mathfrak{A}_i)$ for $i \in I \neq \emptyset$, mappings $f_i : \Omega \to \Omega_i$ for $i \in I$ and some non-empty set $\Omega$.

**Definition 3.** The *$\sigma$-algebra generated by $(f_i)_{i\in I}$ (and $(\mathfrak{A}_i)_{i\in I}$)*

$$\sigma(\{f_i : i \in I\}) = \sigma\Big(\bigcup_{i\in I} f_i^{-1}(\mathfrak{A}_i)\Big).$$

Moreover, set $\sigma(f) = \sigma(\{f\})$.

**Remark 2.** $\sigma(\{f_i : i \in I\})$ is the smallest $\sigma$-algebra $\mathfrak{A}$ in $\Omega$ such that all mappings $f_i$ are $\mathfrak{A}$-$\mathfrak{A}_i$-measurable.

**Theorem 3.** For every measurable space $(\widetilde{\Omega}, \widetilde{\mathfrak{A}})$ and every mapping $g : \widetilde{\Omega} \to \Omega$,

$$g \text{ is } \widetilde{\mathfrak{A}}\text{-}\sigma(\{f_i : i \in I\})\text{-measurable} \quad \Leftrightarrow \quad \forall\, i \in I : f_i \circ g \text{ is } \widetilde{\mathfrak{A}}\text{-}\mathfrak{A}_i\text{-measurable.}$$

*Proof.* Use Lemma 1 to obtain

$$g^{-1}(\sigma(\{f_i : i \in I\})) = \sigma\Big(g^{-1}\Big(\bigcup_{i\in I} f_i^{-1}(\mathfrak{A}_i)\Big)\Big) = \sigma\Big(\bigcup_{i\in I} (f_i \circ g)^{-1}(\mathfrak{A}_i)\Big).$$

Therefore

$$g^{-1}\left(\sigma(\{f_i : i \in I\})\right) \subset \widetilde{\mathfrak{A}} \quad \Leftrightarrow \quad \forall\, i \in I : f_i \circ g_i \text{ is } \widetilde{\mathfrak{A}}\text{-}\mathfrak{A}_i\text{-measurable.}$$

□

Now we turn to the particular case of functions with values in $\mathbb{R}$ or $\overline{\mathbb{R}}$, and we consider the Borel $\sigma$-algebra in $\mathbb{R}$ or $\overline{\mathbb{R}}$, respectively. For any measurable space $(\Omega, \mathfrak{A})$ we use the following notation

$$\begin{aligned}
\mathfrak{Z}(\Omega, \mathfrak{A}) &= \{f : \Omega \to \mathbb{R} : f \text{ is } \mathfrak{A}\text{-}\mathfrak{B}\text{-measurable}\}\,, \\
\mathfrak{Z}_+(\Omega, \mathfrak{A}) &= \{f \in \mathfrak{Z}(\Omega, \mathfrak{A}) : f \geq 0\}\,, \\
\overline{\mathfrak{Z}}(\Omega, \mathfrak{A}) &= \left\{f : \Omega \to \overline{\mathbb{R}} : f \text{ is } \mathfrak{A}\text{-}\overline{\mathfrak{B}}\text{-measurable}\right\}, \\
\overline{\mathfrak{Z}}_+(\Omega, \mathfrak{A}) &= \left\{f \in \overline{\mathfrak{Z}}(\Omega, \mathfrak{A}) : f \geq 0\right\}.
\end{aligned}$$

Every function $f : \Omega \to \mathbb{R}$ may also be considered as a function with values in $\overline{\mathbb{R}}$, and in this case $f \in \mathfrak{Z}(\Omega, \mathfrak{A})$ iff $f \in \overline{\mathfrak{Z}}(\Omega, \mathfrak{A})$.

**Corollary 2.** For $\prec\, \in \{\leq, <, \geq, >\}$ and $f : \Omega \to \overline{\mathbb{R}}$,

$$f \in \overline{\mathfrak{Z}}(\Omega, \mathfrak{A}) \quad \Leftrightarrow \quad \forall\, a \in \mathbb{R} : \{f \prec a\} \in \mathfrak{A}.$$

*Proof.* (Compare Satz 5.6, Bem.5.7, Analysis IV) For instance, $\overline{\mathfrak{B}} = \sigma(\{[-\infty, a] : a \in \mathbb{R}\})$ and

$$\{f \le a\} = f^{-1}([-\infty, a])$$

and $\overline{\mathfrak{B}} = \sigma(\{[-\infty, a] : a \in \mathbb{R}\})$, see Remark 1.6. It remains to apply Theorem 2. □

**Theorem 4.** For $f, g \in \overline{\mathfrak{Z}}(\Omega, \mathfrak{A})$ and $\prec \in \{\le, <, \ge, >, =, \ne\}$,

$$\{\omega \in \Omega : f(\omega) \prec g(\omega)\} \in \mathfrak{A}.$$

*Proof.* For instance, Corollary 2 yields

$$\begin{aligned}\{\omega \in \Omega : f(\omega) < g(\omega)\} &= \bigcup_{q \in \mathbb{Q}} \{f < q < g\} \\ &= \bigcup_{q \in \mathbb{Q}} (\{f < q\} \cap \{g > q\}) \in \mathfrak{A}.\end{aligned}$$

□

**Theorem 5.** For every sequence $f_1, f_2, \ldots \in \overline{\mathfrak{Z}}(\Omega, \mathfrak{A})$,

(i) $\inf_{n \in \mathbb{N}} f_n,\ \sup_{n \in \mathbb{N}} f_n \in \overline{\mathfrak{Z}}(\Omega, \mathfrak{A})$,

(ii) $\liminf_{n \to \infty} f_n,\ \limsup_{n \to \infty} f_n \in \overline{\mathfrak{Z}}(\Omega, \mathfrak{A})$,

(iii) if $(f_n)_{n \in \mathbb{N}}$ converges at every point $\omega \in \Omega$, then $\lim_{n \to \infty} f_n \in \overline{\mathfrak{Z}}(\Omega, \mathfrak{A})$.

*Proof.* (Compare Satz 5.8, 5.9, Analysis IV) For $a \in \mathbb{R}$

$$\left\{\inf_{n \in \mathbb{N}} f_n < a\right\} = \bigcup_{n \in \mathbb{N}} \{f_n < a\}, \quad \left\{\sup_{n \in \mathbb{N}} f_n \le a\right\} = \bigcap_{n \in \mathbb{N}} \{f_n \le a\}.$$

Hence, Corollary 2 yields (i). Since

$$\limsup_{n \to \infty} f_n = \inf_{m \in \mathbb{N}} \sup_{n \ge m} f_n, \quad \liminf_{n \to \infty} f_n = \sup_{m \in \mathbb{N}} \inf_{n \ge m} f_n,$$

we obtain (ii) from (i). Finally, (iii) follows from (ii). □

By

$$f^+ = \max(0, f), \quad f^- = \max(0, -f)$$

we denote the positive part and the negative part, respectively, of $f : \Omega \to \overline{\mathbb{R}}$.

**Remark 3.** For $f \in \overline{\mathfrak{Z}}(\Omega, \mathfrak{A})$ we have $f^+, f^-, |f| \in \overline{\mathfrak{Z}}_+(\Omega, \mathfrak{A})$.

**Theorem 6.** For $f, g \in \overline{\mathfrak{Z}}(\Omega, \mathfrak{A})$,

$$f \pm g,\ f \cdot g,\ f/g \in \overline{\mathfrak{Z}}(\Omega, \mathfrak{A}),$$

provided that these functions are well defined.

*Proof.* (Compare Folgerung 5.5, Analysis IV) The proof is again based on Corollary 2. For simplicity we only consider the case that $f$ and $g$ are real-valued. Clearly $g \in \overline{\mathfrak{Z}}(\Omega, \mathfrak{A})$ implies $-g \in \overline{\mathfrak{Z}}(\Omega, \mathfrak{A})$, too. Furthermore, for every $a \in \mathbb{R}$,

$$\{f + g < a\} = \bigcup_{q \in \mathbb{Q}} \{f < q\} \cap \{g < a - q\},$$

and therefore $f \pm g \in \overline{\mathfrak{Z}}(\Omega, \mathfrak{A})$. Clearly $f \cdot g \in \overline{\mathfrak{Z}}(\Omega, \mathfrak{A})$ if $f$ is constant. Moreover, $x \mapsto x^2$ defines a $\mathfrak{B}$-$\mathfrak{B}$-measurable function, see Corollary 1, and

$$f \cdot g = 1/4 \cdot \big((f+g)^2 - (f-g)^2\big)$$

We apply Theorem 1 to obtain $f \cdot g \in \overline{\mathfrak{Z}}(\Omega, \mathfrak{A})$ in general. Finally, it is easy to show that $g \in \overline{\mathfrak{Z}}(\Omega, \mathfrak{A})$ implies $1/g \in \overline{\mathfrak{Z}}(\Omega, \mathfrak{A})$. □

**Definition 4.** $f \in \mathfrak{Z}(\Omega, \mathfrak{A})$ is called *simple function* if $|f(\Omega)| < \infty$. Put

$$\begin{aligned} \Sigma(\Omega, \mathfrak{A}) &= \{f \in \mathfrak{Z}(\Omega, \mathfrak{A}) : f \text{ simple}\}, \\ \Sigma_+(\Omega, \mathfrak{A}) &= \{f \in \Sigma(\Omega, \mathfrak{A}) : f \geq 0\}. \end{aligned}$$

**Remark 4.** $f \in \Sigma(\Omega, \mathfrak{A})$ iff

$$f = \sum_{i=1}^{n} \alpha_i \cdot 1_{A_i}$$

with $\alpha_1, \dots \alpha_n \in \mathbb{R}$ pairwise different and $A_1, \dots, A_n \in \mathfrak{A}$ pairwise disjoint such that $\bigcup_{i=1}^n A_i = \Omega$.

**Theorem 7.** (Compare Theorem 5.11, Analysis IV) For every (bounded) function $f \in \overline{\mathfrak{Z}}_+(\Omega, \mathfrak{A})$ there exists a sequence $f_1, f_2, \dots \in \Sigma_+(\Omega, \mathfrak{A})$ such that $f_n \uparrow f$ (with uniform convergence).

*Proof.* Let $n \in N$ and put

$$f_n = \sum_{k=1}^{n \cdot 2^n} \frac{k-1}{2^n} \cdot 1_{A_{n,k}} + n \cdot 1_{B_n}$$

where

$$A_{n,k} = \{(k-1)/(2^n) \leq f < k/(2^n)\}, \quad B_n = \{f \geq n\}.$$

□

Now we consider a mapping $T : \Omega_1 \to \Omega_2$ and a $\sigma$-algebra $\mathfrak{A}_2$ in $\Omega_2$. We characterize measurability of functions with respect to $\sigma(T) = T^{-1}(\mathfrak{A}_2)$.

**Theorem 8 (Factorization Lemma).** For every function $f : \Omega_1 \to \overline{\mathbb{R}}$

$$f \in \overline{\mathfrak{Z}}(\Omega_1, \sigma(T)) \quad \Leftrightarrow \quad \exists\, g \in \overline{\mathfrak{Z}}(\Omega_2, \mathfrak{A}_2) : f = g \circ T.$$

*Proof.* '$\Leftarrow$' is trivially satisfied. '$\Rightarrow$': First, assume that $f \in \Sigma_+(\Omega_1, \sigma(T))$, i.e.,

$$f = \sum_{i=1}^{n} \alpha_i \cdot 1_{A_i}$$

with pairwise disjoint sets $A_1, \ldots, A_n \in \sigma(T)$. Take pairwise disjoint sets $B_1, \ldots, B_n \in \mathfrak{A}_2$ such that $A_i = T^{-1}(B_i)$ and put

$$g = \sum_{i=1}^{n} \alpha_i \cdot 1_{B_i}.$$

Clearly $f = g \circ T$ and $g \in \overline{\mathfrak{Z}}(\Omega_2, \mathfrak{A}_2)$.
Now, assume that $f \in \overline{\mathfrak{Z}}_+(\Omega_1, \sigma(T))$. Take a sequence $(f_n)_{n\in\mathbb{N}}$ in $\Sigma_+(\Omega_1, \sigma(T))$ according to Theorem 7. We already know that $f_n = g_n \circ T$ for suitable $g_n \in \overline{\mathfrak{Z}}(\Omega_2, \mathfrak{A}_2)$. Hence

$$f = \sup_n f_n = \sup_n (g_n \circ T) = (\sup_n g_n) \circ T = g \circ T$$

where $g = \sup_n g_n \in \overline{\mathfrak{Z}}(\Omega_2, \mathfrak{A}_2)$.
In the general case, we already know that

$$f^+ = g_1 \circ T, \quad f^- = g_2 \circ T$$

for suitable $g_1, g_2 \in \overline{\mathfrak{Z}}(\Omega_2, \mathfrak{A}_2)$. Put

$$C = \{g_1 = g_2 = \infty\} \in \mathfrak{A}_2,$$

and observe that $T(\Omega_1) \cap C = \emptyset$ since $f = f^+ - f^-$. We conclude that $f = g \circ T$ where

$$g = g_1 \cdot 1_D - g_2 \cdot 1_D \in \overline{\mathfrak{Z}}(\Omega_2, \mathfrak{A}_2)$$

with $D = C^c$. $\square$

Our method of proof for Theorem 8 is sometimes called *algebraic induction.*

# 3 Product Spaces

**Example 1.** A stochastic model for coin tossing. For a single trial,

$$\Omega = \{0, 1\}, \quad \mathfrak{A} = \mathfrak{P}(\Omega), \quad \forall\, \omega \in \Omega : P(\{\omega\}) = 1/2. \tag{1}$$

For $n$ 'independent' trials, (1) serves as a building-block,

$$\Omega_i = \{0, 1\}, \quad \mathfrak{A}_i = \mathfrak{P}(\Omega_i), \quad \forall\, \omega_i \in \Omega_i : P_i(\{\omega_i\}) = 1/2,$$

and we define

$$\Omega = \prod_{i=1}^{n} \Omega_i, \quad \mathfrak{A} = \mathfrak{P}(\Omega), \quad \forall\, A \in \mathfrak{A} : P(A) = \frac{|A|}{|\Omega|}.$$

Then

$$P(A_1 \times \cdots \times A_n) = P_1(A_1) \cdot \cdots \cdot P_n(A_n)$$

for all $A_i \in \mathfrak{A}_i$.

Question: How to model an infinite sequence of trials? To this end,

$$\Omega = \prod_{i=1}^{\infty} \Omega_i.$$

How to choose a $\sigma$-algebra $\mathfrak{A}$ in $\Omega$ and a probability measure $P$ on $(\Omega, \mathfrak{A})$? A reasonable requirement is

$$\begin{aligned} &\forall\, n \in \mathbb{N}\ \forall\, A_i \in \mathfrak{A}_i : \\ &\qquad P(A_1 \times \cdots \times A_n \times \Omega_{n+1} \times \Omega_{n+2} \ldots) = P_1(A_1) \cdot \cdots \cdot P_n(A_n). \end{aligned} \tag{2}$$

Unfortunately,

$$\mathfrak{A} = \mathfrak{P}(\Omega)$$

is too large, since there exists no probability measure on $(\Omega, \mathfrak{P}(\Omega))$ such that (2) holds. The latter fact follows from a theorem by Banach and Kuratowski, which relies on the continuum hypothesis, see Dudley (2002, p. 526). On the other hand,

$$\mathfrak{A} = \{A_1 \times \cdots \times A_n \times \Omega_{n+1} \times \Omega_{n+2} \cdots : n \in \mathbb{N},\ A_i \in \mathfrak{A}_i \text{ for } i = 1, \ldots, n\} \tag{3}$$

is not a $\sigma$-algebra.

Given: a non-empty set $I$ and measurable spaces $(\Omega_i, \mathfrak{A}_i)$ for $i \in I$. Put

$$Y = \bigcup_{i \in I} \Omega_i$$

and define

$$\prod_{i \in I} \Omega_i = \{\omega \in Y^I : \omega(i) \in \Omega_i \text{ for } i \in I\}.$$

Notation: $\omega = (\omega_i)_{i \in I}$ for $\omega \in \prod_{i \in I} \Omega_i$. Moreover, let

$$\mathfrak{P}_0(I) = \{J \subset I : J \text{ non-empty, finite}\}.$$

The following definition is motivated by (3).

**Definition 1.**

(i) *Measurable rectangle*

$$A = \prod_{j \in J} A_j \times \prod_{i \in I \setminus J} \Omega_i$$

with $J \in \mathfrak{P}_0(I)$ and $A_j \in \mathfrak{A}_j$ for $j \in J$. Notation: $\mathfrak{R}$ class of measurable rectangles.

(ii) *Product (measurable) space* $(\Omega, \mathfrak{A})$ with *components* $(\Omega_i, \mathfrak{A}_i)$, $i \in I$,

$$\Omega = \prod_{i \in I} \Omega_i, \qquad \mathfrak{A} = \sigma(\mathfrak{R}).$$

Notation: $\mathfrak{A} = \bigotimes_{i \in I} \mathfrak{A}_i$, *product $\sigma$-algebra*.

**Remark 1.** The class $\mathfrak{R}$ is a semi-algebra, but not an algebra in general. See Übung 2.3.

**Example 2.** Obviously, (2) only makes sense if $\mathfrak{A}$ contains the product $\sigma$-algebra $\bigotimes_{i=1}^n \mathfrak{A}_i$. We will show that there exists a uniquely determined probability measure $P$ on the product space $(\prod_{i=1}^\infty \{0,1\}, \bigotimes_{i=1}^\infty \mathfrak{P}(\{0,1\}))$ that satisfies (2), see Remark 4.3.(ii). The corresponding probability space yields a stochastic model for the simple case of gambling, which was mentioned in the introductory Example I.2.

We study several classes of mappings or subsets that generate the product $\sigma$-algebra. Moreover, we characterize measurability of mappings that take values in a product space.

Put $\Omega = \prod_{i \in I} \Omega_i$. For any $\emptyset \neq S \subset I$ let

$$\pi_S^I : \Omega \to \prod_{i \in S} \Omega_i, \quad (\omega_i)_{i \in I} \mapsto (\omega_i)_{i \in S}$$

denote the *projection* of $\Omega$ onto $\prod_{i \in S} \Omega_i$ (restriction of mappings $\omega$). In particular, for $i \in I$ the $i$-th projection is given by $\pi_{\{i\}}^I$. Sometimes we simply write $\pi_S$ instead of $\pi_S^I$ and $\pi_i$ instead of $\pi_{\{i\}}$.

**Theorem 1.**

(i) $\bigotimes_{i\in I}\mathfrak{A}_i = \sigma(\{\pi_i : i \in I\})$.

(ii) $\forall\, i \in I : \mathfrak{A}_i = \sigma(\mathfrak{E}_i) \quad \Rightarrow \quad \bigotimes_{i\in I}\mathfrak{A}_i = \sigma\left(\bigcup_{i\in I}\pi_i^{-1}(\mathfrak{E}_i)\right)$.

*Proof.* Ad (i), '$\supset$': We show that every projection $\pi_i : \Omega \to \Omega_i$ is $\left(\bigotimes_{i\in I}\mathfrak{A}_i\right)$-$\mathfrak{A}_i$-measurable. For $A_i \in \mathfrak{A}_i$

$$\pi_i^{-1}(A_i) = A_i \times \prod_{i\in I\setminus\{i\}}\Omega_i \in \mathfrak{R}.$$

Ad (i), '$\subset$': We show that $\mathfrak{R} \subset \sigma(\{\pi_i : i \in I\})$. For $J \in \mathfrak{P}_0(I)$ and $A_j \in \mathfrak{A}_j$ with $j \in J$

$$\prod_{j\in J}A_j \times \prod_{i\in I\setminus J}\Omega_i = \bigcap_{j\in J}\pi_j^{-1}(A_j).$$

Ad (ii): By Lemma 2.1 and (i)

$$\bigotimes_{i\in I}\mathfrak{A}_i = \sigma\Big(\bigcup_{i\in I}\pi_i^{-1}(\mathfrak{A}_i)\Big) = \sigma\Big(\bigcup_{i\in I}\sigma(\pi_i^{-1}(\mathfrak{E}_i))\Big) = \sigma\Big(\bigcup_{i\in I}\pi_i^{-1}(\mathfrak{E}_{\mathrm{i}})\Big).$$

□

**Corollary 1.**

(i) For every measurable space $(\widetilde{\Omega}, \widetilde{\mathfrak{A}})$ and every mapping $g : \widetilde{\Omega} \to \Omega$

$$g \text{ is } \widetilde{\mathfrak{A}}\text{-}\bigotimes_{i\in I}\mathfrak{A}_i\text{-measurable} \quad \Leftrightarrow \quad \forall\, i \in I : \pi_i \circ g \text{ is } \widetilde{\mathfrak{A}}\text{-}\mathfrak{A}_i\text{-measurable.}$$

(ii) For every $\emptyset \neq S \subset I$ the projection $\pi_S^I$ is $\bigotimes_{i\in I}\mathfrak{A}_i$-$\bigotimes_{i\in S}\mathfrak{A}_i$-measurable.

*Proof.* Ad (i): Follows immediately from Theorem 2.3 and Theorem 1.(i).
Ad (ii): Note that $\pi_{\{i\}}^S \circ \pi_S^I = \pi_i^I$ and use (i). □

**Remark 2.** From Theorem 1.(i) and Corollary 1 we get

$$\bigotimes_{i\in I}\mathfrak{A}_i = \sigma(\{\pi_S^I : S \in \mathfrak{P}_0(I)\}).$$

The sets

$$\left(\pi_S^I\right)^{-1}(B) = B \times \Big(\prod_{i\in I\setminus S}\Omega_i\Big)$$

with $S \in \mathfrak{P}_0(I)$ and $B \in \bigotimes_{i\in S}\mathfrak{A}_i$ are called *cylinder sets*. Notation: $\mathfrak{C}$ class of cylinder sets. The class $\mathfrak{C}$ is an algebra in $\Omega$, but not a $\sigma$-algebra in general. Moreover,

$$\mathfrak{R} \subset \alpha(\mathfrak{R}) \subset \mathfrak{C} \subset \sigma(\mathfrak{R}),$$

where equality does not hold in general.

Every product measurable set is countably determined in the following sense.

**Theorem 2.** For every $A \in \otimes_{i\in I}\mathfrak{A}_i$ there exists a non-empty countable set $S \subset I$ and a set $B \in \otimes_{i\in S}\mathfrak{A}_i$ such that

$$A = \left(\pi_S^I\right)^{-1}(B).$$

*Proof.* Put

$$\widetilde{\mathfrak{A}} = \Big\{A \in \bigotimes_{i\in I}\mathfrak{A}_i : \exists\, S \subset I \text{ non-empty, countable } \exists\, B \in \bigotimes_{i\in S}\mathfrak{A}_i : A = \left(\pi_S^I\right)^{-1}(B)\Big\}.$$

By definition, $\widetilde{\mathfrak{A}}$ contains every cylinder set and $\widetilde{\mathfrak{A}} \subset \bigotimes_{i\in I}\mathfrak{A}_i$. It remains to show that $\widetilde{\mathfrak{A}}$ is a $\sigma$-algebra. Obviously, $\Omega \in \widetilde{\mathfrak{A}}$, and if $A = (\pi_S^I)^{-1}(B)$, $A^c = (\pi_S^I)^{-1}(B^c)$. Finally, if $A_n = (\pi_{S_n}^I)^{-1}(B_n)$, we define $S = \bigcup_n S_n$ and $\widetilde{B}_n = (\pi_{S_n}^S)^{-1}(B_n) = B_n \times \prod_{i\in S\setminus B_n} \in \bigotimes_{i\in S}\mathfrak{A}_i$ (see Corollary 1, (ii)); then

$$\bigcap_n A_n = \bigcap_n (\pi_S^I)^{-1}(\widetilde{B}_n) = ((\pi^I)_S)^{-1}\left(\bigcap_n \widetilde{B}_n\right),$$

hence $\bigcap_n A_n \in \widetilde{\mathfrak{A}}$. □

Now we study products of Borel-$\sigma$-algebras.

**Theorem 3.**

$$\mathfrak{B}_k = \bigotimes_{i=1}^{k}\mathfrak{B}, \qquad \overline{\mathfrak{B}}_k = \bigotimes_{i=1}^{k}\overline{\mathfrak{B}}.$$

*Proof.* By Remark 16,

$$B_k = \sigma\Big(\Big\{\prod_{i=1}^{k}]-\infty, a_i] : a_i \in \mathbb{R} \text{ for } i = 1, \dots, k\Big\}\Big) \subset \bigotimes_{i=1}^{k}\mathfrak{B}.$$

On the other hand, $\pi_i : \mathbb{R}^k \to \mathbb{R}$ is continuous, hence it remains to apply Corollary 2.1 and Theorem 1.(i). Analogously, $\overline{\mathfrak{B}}_k = \bigotimes_{i=1}^{k}\overline{\mathfrak{B}}$ follows. □

**Remark 3.** Consider a measurable space $(\widetilde{\Omega}, \widetilde{\mathfrak{A}})$ and a mapping

$$f = (f_1, \dots, f_k) : \widetilde{\Omega} \to \overline{\mathbb{R}}^k.$$

Then, according to Theorem 3, $f$ is $\widetilde{\mathfrak{A}}$-$\overline{\mathfrak{B}}_k$-measurable iff all functions $f_i$ are $\widetilde{\mathfrak{A}}$-$\overline{\mathfrak{B}}$-measurable.

# 4 Construction of (Probability) Measures

Given: $\Omega \neq \emptyset$ and $\emptyset \neq \mathfrak{A} \subset \mathfrak{P}(\Omega)$.

**Definition 1.** $\mu : \mathfrak{A} \to \mathbb{R}_+ \cup \{\infty\}$ is called

(i) *additive* if:

$$A, B \in \mathfrak{A} \wedge A \cap B = \emptyset \wedge A \cup B \in \mathfrak{A} \quad \Rightarrow \quad \mu(A \cup B) = \mu(A) + \mu(B),$$

(ii) $\sigma$*-additive* if

$$A_1, A_2, \ldots \in \mathfrak{A} \text{ pairwise disjoint } \wedge \bigcup_{i=1}^{\infty} A_i \in \mathfrak{A} \quad \Rightarrow \quad \mu\Big(\bigcup_{i=1}^{\infty} A_i\Big) = \sum_{i=1}^{\infty} \mu(A_i),$$

(iii) *content (on* $\mathfrak{A}$*)* if

$$\mathfrak{A} \text{ algebra} \quad \wedge \quad \mu \text{ additive} \quad \wedge \quad \mu(\emptyset) = 0,$$

(iv) *pre-measure (on* $\mathfrak{A}$*)* if

$$\mathfrak{A} \text{ semi-algebra} \quad \wedge \quad \mu\ \sigma\text{-additive} \quad \wedge \quad \mu(\emptyset) = 0,$$

(v) *measure (on* $\mathfrak{A}$*)* if

$$\mathfrak{A}\ \sigma\text{-algebra} \quad \wedge \quad \mu \text{ pre-measure},$$

(vi) *probability measure (on* $\mathfrak{A}$*)* if

$$\mu \text{ measure} \quad \wedge \quad \mu(\Omega) = 1.$$

**Definition 2.** $(\Omega, \mathfrak{A}, \mu)$ is called a

(i) *measure space*, if $\mu$ is a measure on the $\sigma$-algebra $\mathfrak{A}$ in $\Omega$,

(ii) *probability space*, if $\mu$ is a probability measure on the $\sigma$-algebra $\mathfrak{A}$ in $\Omega$.

**Example 1.**

(i) *k–dimensional Lebesgue pre-measure* $\lambda_k$, e.g., on cartesian products of intervals.

(ii) For any semi-algebra $\mathfrak{A}$ in $\Omega$ and $\omega \in \Omega$

$$\delta_\omega(A) = 1_A(\omega), \qquad A \in \mathfrak{A},$$

defines a pre-measure. If $\mathfrak{A}$ is a $\sigma$-algebra, then $\delta_\omega$ is called the *Dirac measure* at the point $\omega$.

More generally: take sequences $(\omega_n)_{n\in\mathbb{R}}$ in $\Omega$ and $(\alpha_n)_{n\in\mathbb{N}}$ in $\mathbb{R}_+$ such that $\sum_{n=1}^{\infty} \alpha_n = 1$. Then

$$\mu(A) = \sum_{n=1}^{\infty} \alpha_n \cdot 1_A(\omega_n), \qquad A \in \mathfrak{A},$$

defines a *discrete probability measure* on any $\sigma$-algebra $\mathfrak{A}$ in $\Omega$. Note that $\mu = \sum_{n=1}^{\infty} \alpha_n \cdot \varepsilon_{\omega_n}$.

(iii) *Counting measure* on a $\sigma$-algebra $\mathfrak{A}$

$$\mu(A) = |A|, \qquad A \in \mathfrak{A}.$$

*Uniform distribution* in the case $|\Omega| < \infty$ and $\mathfrak{A} = \mathfrak{P}(\Omega)$

$$\mu(A) = \frac{|A|}{|\Omega|}, \qquad A \subset \Omega.$$

(iv) On the algebra $\mathfrak{A} = \{A \subset \Omega : A \text{ finite or } A^c \text{ finite}\}$ let

$$\mu(A) = \begin{cases} 0 & \text{if } |A| < \infty \\ \infty & \text{if } |A| = \infty. \end{cases}$$

Then $\mu$ is a content but not a pre-measure in general.

(v) For the semi-algebra of measurable rectangles in Example 3.1 and $A_i \subset \{0,1\}$

$$\mu(A_1 \times \cdots \times A_n \times \Omega_{n+1} \times \cdots) = \frac{|A_1 \times \ldots \times A_n|}{|\{0,1\}^n|}$$

is well defined and yields a pre-measure $\mu$ with $\mu\big(\{0,1\}^{\mathbb{N}}\big) = 1$.

**Remark 1.** For every content $\mu$ on $\mathfrak{A}$ and $A, B \in \mathfrak{A}$

(i) $A \subset B \Rightarrow \mu(A) \le \mu(A \cap B) + \mu(A^c \cap B) = \mu(B)$ (*monotonicity*),

(ii) $\mu(A \cup B) + \mu(A \cap B) = \mu(A) + \mu(B \setminus A) + \mu(A \cap B) = \mu(A) + \mu(B)$,

(iii) $A \subset B \wedge \mu(A) < \infty \Rightarrow \mu(B \setminus A) = \mu(B) - \mu(A)$,

(iv) $\mu(A) < \infty \wedge \mu(B) < \infty \Rightarrow |\mu(A) - \mu(B)| \le \mu(A \triangle B)$,

(v) $\mu(A \cup B) = \mu(A) + \mu(B \cap A^c) \le \mu(A) + \mu(B)$ (*subadditivity*).

**Theorem 1.** Consider the following properties for a content $\mu$ on $\mathfrak{A}$:

(i) $\mu$ pre-measure,

(ii) $A_1, A_2, \ldots \in \mathfrak{A} \wedge \bigcup_{i=1}^\infty A_i \in \mathfrak{A} \Rightarrow \mu\big(\bigcup_{i=1}^\infty A_i\big) \le \sum_{i=1}^\infty \mu(A_i)$ ($\sigma$-*subadditivity*),

(iii) $A_1, A_2, \ldots \in \mathfrak{A} \wedge A_n \uparrow A \in \mathfrak{A} \Rightarrow \lim_{n\to\infty} \mu(A_n) = \mu(A)$ ($\sigma$-*continuity from below*),

(iv) $A_1, A_2, \ldots \in \mathfrak{A} \wedge A_n \downarrow A \in \mathfrak{A} \wedge \mu(A_1) < \infty \Rightarrow \lim_{n\to\infty} \mu(A_n) = \mu(A)$ ($\sigma$-*continuity from above*),

(v) $A_1, A_2, \ldots \in \mathfrak{A} \wedge A_n \downarrow \emptyset \wedge \mu(A_1) < \infty \Rightarrow \lim_{n\to\infty} \mu(A_n) = 0$ ($\sigma$-*continuity at* $\emptyset$).

Then

$$\text{(i)} \Leftrightarrow \text{(ii)} \Leftrightarrow \text{(iii)} \Rightarrow \text{(iv)} \Leftrightarrow \text{(v)}.$$

If $\mu(\Omega) < \infty$, then (iii) $\Leftrightarrow$ (iv).

*Proof.* '(i) $\Rightarrow$ (ii)': Put $B_m = \bigcup_{i=1}^m A_i$ and $B_0 = \emptyset$. Then

$$\bigcup_{i=1}^\infty A_i = \bigcup_{m=1}^\infty (B_m \setminus B_{m-1})$$

with pairwise disjoint sets $B_m \setminus B_{m-1} \in \mathfrak{A}$. Clearly $B_m \setminus B_{m-1} \subset A_m$. Hence, by Remark 1.(i),

$$\mu\Big(\bigcup_{i=1}^\infty A_i\Big) = \sum_{m=1}^\infty \mu(B_m \setminus B_{m-1}) \leq \sum_{m=1}^\infty \mu(A_m).$$

'(ii) $\Rightarrow$ (i)': Let $A_1, A_2, \ldots \in \mathfrak{A}$ be pairwise disjoint with $\bigcup_{i=1}^\infty A_i \in \mathfrak{A}$. Then

$$\mu\Big(\bigcup_{i=1}^\infty A_i\Big) \geq \mu\Big(\bigcup_{i=1}^n A_i\Big) = \sum_{i=1}^n \mu(A_i),$$

and therefore

$$\sum_{i=1}^\infty \mu(A_i) \leq \mu\Big(\bigcup_{i=1}^\infty A_i\Big).$$

The reverse estimate holds by assumption.
'(i) $\Rightarrow$ (iii)': Put $A_0 = \emptyset$ and $B_m = A_m \setminus A_{m-1}$. Then

$$\mu\Big(\bigcup_{i=1}^\infty A_i\Big) = \sum_{m=1}^\infty \mu(B_m) = \lim_{n\to\infty} \sum_{m=1}^n \mu(B_m) = \lim_{n\to\infty} \mu\Big(\bigcup_{m=1}^n B_m\Big) = \lim_{n\to\infty} \mu(A_n).$$

'(iii) $\Rightarrow$ (i)': Let $A_1, A_2, \ldots \in \mathfrak{A}$ be pairwise disjoint with $\bigcup_{i=1}^\infty A_i \in \mathfrak{A}$, and put $B_m = \bigcup_{i=1}^m A_i$. Then $B_m \uparrow \bigcup_{i=1}^\infty A_i$ and

$$\mu\Big(\bigcup_{i=1}^\infty A_i\Big) = \lim_{m\to\infty} \mu(B_m) = \sum_{i=1}^\infty \mu(A_i).$$

'(iv) $\Rightarrow$ (v)' trivially holds.
'(v) $\Rightarrow$ (iv)': Use $B_n = A_n \setminus A \downarrow \emptyset$.
'(i)' $\Rightarrow$ (v)': Note that $\mu(A_1) = \sum_{i=1}^\infty \mu(A_i \setminus A_{i+1})$. Hence

$$0 = \lim_{k\to\infty} \sum_{i=k}^\infty \mu(A_i \setminus A_{i+1}) = \lim_{k\to\infty} \mu(A_k).$$

'(iv) $\wedge$ $\mu(\Omega) < \infty \Rightarrow$ (iii)': Clearly $A_n \uparrow A$ implies $A_n^c \downarrow A^c$. Thus

$$\mu(A) = \mu(\Omega) - \mu(A^c) = \lim_{n\to\infty} (\mu(\Omega) - \mu(A_n^c)) = \lim_{n\to\infty} \mu(A_n).$$

$\square$

**Theorem 2 (Extension: semi-algebra $\rightsquigarrow$ algebra).** For every semi-algebra $\mathfrak{A}$ and every additive mapping $\mu : \mathfrak{A} \to \mathbb{R}_+ \cup \{\infty\}$ with $\mu(\emptyset) = 0$

$$\underset{1}{\exists}\, \widehat{\mu} \text{ content on } \alpha(\mathfrak{A}) : \quad \widehat{\mu}|_{\mathfrak{A}} = \mu.$$

Moreover, if $\mu$ is $\sigma$-additive then $\widehat{\mu}$ is $\sigma$-additive, too.

*Proof.* We have $\alpha(\mathfrak{A}) = \mathfrak{A}^+$, see Lemma 1.1. Necessarily

$$\widehat{\mu}\Big(\bigcup_{i=1}^{n} A_i\Big) = \sum_{i=1}^{n} \mu(A_i) \tag{1}$$

for $A_1, \ldots, A_n \in \mathfrak{A}$ pairwise disjoint. Use (1) to obtain a well-defined extension of $\mu$ onto $\alpha(\mathfrak{A})$. It easily follows that $\mu$ is additive or even $\sigma$-additive. □

**Example 2.** For the semi-algebra $\mathfrak{A}$ in Example 1.(v) $\alpha(\mathfrak{A})$ is the algebra of cylinder sets, and

$$\widehat{\mu}(A \times \Omega_{n+1} \times \cdots) = \frac{|A|}{|\{0,1\}^n|}, \qquad A \subset \{0,1\}^n .$$

Let $\mu$ be a pre–measure on $\mathfrak{A}$. The *outer measure* generated by $\mu$ is

$$\mu^*(A) := \inf\left\{ \sum_{i=1}^{\infty} \mu(A_i) \ : \ A_i \in \mathfrak{A}, A \subseteq \bigcup_{i=1} \infty A_i \right\} ,$$

It is straightforward that $\mu^*(\emptyset = 0)$ and that $\mu^*$ is monotone and $\sigma$–subadditive.

**Theorem 3 (Extension: algebra $\rightsquigarrow$ $\sigma$-algebra, Carathéodory).** For every pre-measure $\mu$ on an algebra $\mathfrak{A}$,

(a) the class

$$\mathfrak{A}_{\mu^*} := \Big\{ A \subseteq \Omega \ : \ \mu^*(B) = \mu^*(A \cap B) + \mu^*(A^c \cap B) \forall B \subseteq \Omega \Big\}$$

is a $\sigma$–algebra, and $\mu^*$ is a measure on $\mathfrak{A}_{\mu^*}$.

(b) $\mathfrak{A} \subseteq \mathfrak{A}_{\mu^*}$, and $\mu = \mu^*$ on $\mathfrak{A}$. In particular, there exists a measure $\mu^*$ on $\sigma(\mathfrak{A})$ extending $\mu$.

*Proof.* We will start with part (b), i.e., we show that

(i) $\mu^*|_{\mathfrak{A}} = \mu$,

(ii) $\forall\, A \in \mathfrak{A}\ \forall\, B \in \mathfrak{P}(\Omega): \quad \mu^*(B) = \mu^*(B \cap A) + \mu^*(B \cap A^c)$.

Ad (i): For $A \in \mathfrak{A}$

$$\mu^*(A) \leq \mu(A) + \sum_{i=2}^{\infty} \mu(\emptyset) = \mu(A),$$

and for $A_i \in \mathfrak{A}$ with $A \subset \bigcup_{i=1}^{\infty} A_i$

$$\mu(A) = \mu\Big(\bigcup_{i=1}^{\infty} (A_i \cap A)\Big) \leq \sum_{i=1}^{\infty} \mu(A_i \cap A) \leq \sum_{i=1}^{\infty} \mu(A_i)$$

follows from Theorem 1.(ii).

Ad (ii): '$\leq$' holds due to sub-additivity of $\mu^*$; if

$$B \subseteq \bigcup_{i=1}^{\infty} A_i$$

with $A_i \in \mathfrak{A}$, then $A_i \cap A, A_i \cap A^c \in \mathfrak{A}$ and

$$B \cap A \subseteq \bigcup_{i=1}^{\infty} A_i \cap A, \qquad B \cap A^c \subseteq \bigcup_{i=1}^{\infty} A_i \cap A^c .$$

This directly implies '$\geq$'.

Now we prove (a); to this end, we claim first that

(iii) $\mathfrak{A}_{\mu^*}$ is $\cap$–closed, $\forall\, A_1, A_2 \in \mathfrak{A}_{\mu^*}\ \forall\, B \in \mathfrak{P}(\Omega): \quad \mu^*(B) = \mu^*(B \cap (A_1 \cap A_2)) + \mu^*(B \cap (A_1 \cap A_2)^c)$.

(iv) $\mathfrak{A}_{\mu^*}$ $^c$–closed,

i.e., $\mathfrak{A}$ is an algebra.
Ad (iii): We have

$$\begin{aligned} \mu^*(B) &= \mu^*(B \cap A_1) + \mu^*(B \cap A_1^c) \\ &= \mu^*(B \cap A_1 \cap A_2) + \mu^*(B \cap A_1 \cap A_2^c) + \mu^*(B \cap A_1^c) \end{aligned}$$

and

$$\mu^*(B \cap (A_1 \cap A_2)^c) = \mu^*(B \cap A_1^c \cup B \cap A_2^c) = \mu^*(B \cap A_2^c \cap A_1) + \mu^*(B \cap A_1^c).$$

Ad (iv): Obvious.
Next we claim that $\mu^*$ is additive on $\mathfrak{A}^*$, and even more,

(v) $\forall\, A_1, A_2 \in \mathfrak{A}_{\mu^*}$ disjoint $\forall\, B \in \mathfrak{P}(\Omega): \quad \mu^*(B \cap (A_1 \cup A_2)) = \mu^*(B \cap A_1) + \mu^*(B \cap A_2)$.

In fact, since $A_1 \cap A_2 = \emptyset$,

$$\mu^*(B \cap (A_1 \cup A_2)) = \mu^*(B \cap A_1) + \mu^*(B \cap A_2 \cap A_1^c) = \mu^*(B \cap A_1) + \mu^*(B \cap A_2).$$

At last, we claim that $\mathfrak{A}^*$ is a Dynkin class and $\mu^*$ is $\sigma$–additive on $\mathfrak{A}^*$, i.e.,

(vi) $\forall\, A_1, A_2, \ldots \in \mathfrak{A}_{\mu^*}$ pairwise disjoint

$$\bigcup_{i=1}^{\infty} A_i \in \mathfrak{A}_{\mu^*} \quad \wedge \quad \mu^*\Big(\bigcup_{i=1}^{\infty} A_i\Big) = \sum_{i=1}^{\infty} \mu^*(A_i).$$

Let $B \in \mathfrak{P}(\Omega)$. By (iv), (v), and monotonicity of $\mu^*$

$$\begin{aligned}
\mu^*(B) &= \mu^*\Big(B \cap \bigcup_{i=1}^n A_i\Big) + \mu^*\Big(B \cap \Big(\bigcup_{i=1}^n A_i\Big)^c\Big) \\
&\geq \sum_{i=1}^n \mu^*(B \cap A_i) + \mu^*\Big(B \cap \Big(\bigcup_{i=1}^\infty A_i\Big)^c\Big).
\end{aligned}$$

Use $\sigma$-subadditivity of $\mu^*$ to get

$$\begin{aligned}
\mu^*(B) &\geq \sum_{i=1}^\infty \mu^*(B \cap A_i) + \mu^*\Big(B \cap \Big(\bigcup_{i=1}^\infty A_i\Big)^c\Big) \\
&\geq \mu^*\Big(B \cap \bigcup_{i=1}^\infty A_i\Big) + \mu^*\Big(B \cap \Big(\bigcup_{i=1}^\infty A_i\Big)^c\Big) \\
&\geq \mu^*(B).
\end{aligned}$$

Hence $\bigcup_{i=1}^\infty A_i \in \mathfrak{A}_{\mu^*}$. Take $B = \bigcup_{i=1}^\infty A_i$ to obtain $\sigma$-additivity of $\mu^*|_{\mathfrak{A}_{\mu^*}}$.
Conclusions:

- $\mathfrak{A}_{\mu^*}$ is a Dynkin class and $\cap$–closed ((iv), (vi)), and hence a $\sigma$-algebra, see Theorem 1.1.(ii),
- $\mathfrak{A} \subset \mathfrak{A}_{\mu^*}$ by (ii), hence $\sigma(\mathfrak{A}) \subset \mathfrak{A}_{\mu^*}$.
- $\mu^*|_{\mathfrak{A}_{\mu^*}}$ is a measure with $\mu^*|_{\mathfrak{A}} = \mu$, see (vi) and (i).

□

**Remark 2.** The extension from Theorem 3 is non-unique, in general. For instance, on $\Omega = \mathbb{R}$, the pre–measure

$$\mu(A) = \infty \cdot \#A = \begin{cases} 0 & \text{if } A = \emptyset \\ \infty & \text{otherwise} \end{cases}, \qquad A \in \alpha(\mathfrak{J}_1)$$

on the algebra generated by intervals (see Ex.1) has the extensions $\mu_1(A) = \#A$ (counting measure) and $\mu_2(A) = \infty \cdot \#A$ to $\mathfrak{B}$.

**Definition 3.** $\mu : \mathfrak{A} \to \mathbb{R}_+ \cup \{\infty\}$ is called

(i) *$\sigma$-finite*, if

$$\exists\, B_1, B_2, \ldots \in \mathfrak{A} \text{ pairwise disjoint}: \quad \Omega = \bigcup_{i=1}^\infty B_i \wedge \forall\, i \in \mathbb{N} : \mu(B_i) < \infty,$$

(ii) *finite*, if $\Omega \in \mathfrak{A}$ and $\mu(\Omega) < \infty$.

**Theorem 4 (Uniqueness).** $\mathfrak{A}_0$ be $\cap$–closed, $\mu_1$, $\mu_2$ be measures on $\mathfrak{A} = \sigma(\mathfrak{A}_0)$. If $\mu_1|_{\mathfrak{A}_0}$ is $\sigma$–finite and $\mu_1|_{\mathfrak{A}_0} = \mu_2|_{\mathfrak{A}_0}$, then $\mu_1 = \mu_2$.

*Proof.* Take $B_i$ according to Definition 3, with $\mathfrak{A}_0$ instead of $\mathfrak{A}$, and put

$$\mathfrak{D}_i = \{A \in \mathfrak{A} : \mu_1(A \cap B_i) = \mu_2(A \cap B_i)\}.$$

Obviously, $\mathfrak{D}_i$ is a Dynkin class and $\mathfrak{A}_0 \subset \mathfrak{D}_i$. Theorem 1.2.(i) yields

$$\mathfrak{D}_i \subset \mathfrak{A} = \sigma(\mathfrak{A}_0) = \delta(\mathfrak{A}_0) \subset \mathfrak{D}_i.$$

Thus $\mathfrak{A} = \mathfrak{D}_i$ and for $A \in \mathfrak{A}$,

$$\mu_1(A) = \sum_{i=1}^{\infty} \mu_1(A \cap B_i) = \sum_{i=1}^{\infty} \mu_2(A \cap B_i) = \mu_2(A).$$

□

**Corollary 1.** For every semi-algebra $\mathfrak{A}$ and every pre-measure $\mu$ on $\mathfrak{A}$ that is $\sigma$-finite

$$\underset{1}{\exists}\, \mu^* \text{ measure on } \sigma(\mathfrak{A}) : \quad \mu^*|_{\mathfrak{A}} = \mu.$$

*Proof.* Use Theorems 2, 3, and 4. □

**Remark 3.** Applications of Corollary 1:

(i) For $\Omega = \mathbb{R}^k$ and the Lebesgue pre-measure $\lambda_k$ on $\mathfrak{J}_k$ we get the Lebesgue measure on $\mathfrak{B}_k$. Notation for the latter: $\lambda_k$.

(ii) In Example 1.(v) there exists a uniquely determined probability measure $P$ on $\bigotimes_{i=1}^{\infty} \mathfrak{P}(\{0,1\})$ such that

$$P(A_1 \times \cdots \times A_n \times \{0,1\} \times \ldots) = \frac{|A_1 \times \cdots \times A_n|}{|\{0,1\}^n|}$$

for $A_1, \ldots, A_n \subset \{0,1\}$. We will study the general construction of product measures in Section 8.

# 5 Integration

For the proofs, see Analysis IV or Elstrodt (1996, Kap. VI).
Fixed in this section: A measure space $(\Omega, \mathfrak{A}, \mu)$. Notation:

- $\Sigma_+ = \Sigma_+(\Omega, \mathfrak{A})$ (nonnegative simple functions),
- $\overline{\mathfrak{Z}}_+ = \overline{\mathfrak{Z}}_+(\Omega, \mathfrak{A})$ (nonnegative $\mathfrak{A}$-$\overline{\mathfrak{B}}$-measurable functions),

**Definition 1.** *Integral* Let $f \in \Sigma_+$,

$$f = \sum_{i=1}^{n} \alpha_i \cdot 1_{A_i} \ , \qquad \alpha_i \in \mathfrak{R}, A_i \in \mathfrak{A} \ .$$

Then define its *Integral* w.r.t. $\mu$ as

$$\int f \, d\mu = \sum_{i=1}^{n} \alpha_i \cdot \mu(A_i) \ .$$

**Lemma 1.** The mapping $\int \cdot \mathrm{d}\mu : \Sigma_+ \to \mathfrak{R}_+$ is

(i) positive–linear: $\int (\alpha f + \beta g) \, d\mu = \alpha \int f \, d\mu + \beta \int g \, d\mu$, $f, g \in \Sigma_+$, $\alpha, \beta \in \mathfrak{R}_+$,

(ii) monotone: $f \leq g \Rightarrow \int f \, d\mu \leq \int g \, d\mu$ (*monotonicity*).

**Definition 2.** *Integral* of $f \in \overline{\mathfrak{Z}}_+$ w.r.t. $\mu$

$$\int f \, d\mu = \sup \Big\{ \int g \, d\mu : g \in \Sigma_+ \wedge g \leq f \Big\}.$$

**Theorem 1 (Monotone convergence, Beppo Levi).** (e.g., Thm.6.4, Analysis IV, SS06) Let $f_n \in \overline{\mathfrak{Z}}_+$ such that

$$\forall \, n \in \mathbb{N} : f_n \leq f_{n+1}.$$

Then

$$\int \sup_n f_n \, d\mu = \sup_n \int f_n \, d\mu.$$

**Remark 1.** For every $f \in \overline{\mathfrak{Z}}_+$ there exists a sequence of functions $f_n \in \Sigma_+$ such that $f_n \uparrow f$, see Theorem 2.7.

**Example 1.** Consider

$$f_n = \frac{1}{n} \cdot 1_{[0,n]}$$

on $(\mathbb{R}, \mathfrak{B}, \lambda_1)$. Then

$$\int f_n \, d\lambda_1 = 1, \qquad \lim_{n \to \infty} f_n = 0.$$

**Lemma 2.** The mapping $\int \cdot \, \mathrm{d}\mu \ : \ \mathfrak{Z}_+ \to \overline{\mathfrak{R}}_+$ is still positive–linear and monotone.

**Theorem 2 (Fatou's Lemma).** (See, e.g., Lemma 6.6, Ananlysis IV, SS06) For every sequence $(f_n)_n$ in $\overline{\mathfrak{Z}}_+$

$$\int \liminf_{n\to\infty} f_n \, d\mu \leq \liminf_{n\to\infty} \int f_n \, d\mu.$$

*Proof.* For $g_n = \inf_{k\geq n} f_k$ we have $g_n \in \overline{\mathfrak{Z}}_+$ and $g_n \uparrow \liminf_n f_n$. By Theorem 1 and Lemma 1.(iii)

$$\int \liminf_n f_n \, d\mu = \lim_{n\to\infty} \int g_n \, d\mu \leq \liminf_{n\to\infty} \int f_n \, d\mu.$$

□

**Theorem 3.** Let $f \in \overline{\mathfrak{Z}}_+$. Then

$$\int f \, d\mu = 0 \Leftrightarrow \mu(\{f > 0\}) = 0.$$

**Definition 3.** A property $\Pi$ holds *$\mu$-almost everywhere ($\mu$-a.e., a.e.)*, if

$$\exists\, A \in \mathfrak{A} : \{\omega \in \Omega : \Pi \text{ does not hold for } \omega\} \subset A \wedge \mu(A) = 0.$$

In case of a probability measure we say: *$\mu$-almost surely, $\mu$-a.s., with probability one*.

Notation: $\overline{\mathfrak{Z}} = \overline{\mathfrak{Z}}(\Omega, \mathfrak{A})$ is the class of $\mathfrak{A}$-$\overline{\mathfrak{B}}$-measurable functions.

**Definition 4.** $f \in \overline{\mathfrak{Z}}$ *quasi-$\mu$-integrable* if

$$\int f_+ \, d\mu < \infty \quad \vee \quad \int f_- \, d\mu < \infty.$$

In this case: *integral* of $f$ (w.r.t. $\mu$)

$$\int f \, d\mu = \int f_+ \, d\mu - \int f_- \, d\mu.$$

$f \in \overline{\mathfrak{Z}}$ *$\mu$-integrable* if

$$\int f_+ \, d\mu < \infty \quad \wedge \quad \int f_- \, d\mu < \infty.$$

**Theorem 4.**

(i) $f$ $\mu$-integrable $\Rightarrow \mu(\{|f| = \infty\}) = 0$,

(ii) $f$ $\mu$-integrable $\wedge\, g \in \overline{\mathfrak{Z}} \wedge f = g$ $\mu$-a.e. $\Rightarrow g$ $\mu$-integrable $\wedge \int f \, d\mu = \int g \, d\mu$.

(iii) equivalent properties for $f \in \overline{\mathfrak{Z}}$:

- (a) $f$ $\mu$-integrable,
- (b) $|f|$ $\mu$-integrable,
- (c) $\exists\, g : g$ $\mu$-integrable $\wedge\, |f| \leq g$ $\mu$-a.e.,

(iv) for $f$ and $g$ $\mu$-integrable and $c \in \mathbb{R}$

  (a) $f+g$ well-defined $\mu$-a.e. and $\mu$-integrable with $\int(f+g)\,d\mu = \int f\,d\mu + \int g\,d\mu$,
  (b) $c \cdot f$ $\mu$-integrable with $\int(cf)\,d\mu = c \cdot \int f\,d\mu$,
  (c) $f \leq g$ $\mu$-a.e. $\Rightarrow \int f\,d\mu \leq \int g\,d\mu$.

**Theorem 5 (Dominated convergence, Lebesgue).** Assume that

(i) $f_n \in \overline{\mathfrak{Z}}$ for $n \in \mathbb{N}$,

(ii) $\exists\, g$ $\mu$-integrable $\forall\, n \in \mathbb{N} : |f_n| \leq g$ $\mu$-a.e.,

(iii) $f \in \overline{\mathfrak{Z}}$ such that $\lim_{n\to\infty} f_n = f$ $\mu$-a.e.

Then $f$ is $\mu$-integrable and

$$\int f\,d\mu = \lim_{n\to\infty} \int f_n\,d\mu.$$

**Example 2.** Consider

$$f_n = n \cdot 1_{]0,1/n[}$$

on $(\mathbb{R}, \mathfrak{B}, \lambda_1)$. Then

$$\int f_n\,d\lambda_1 = 1, \qquad \lim_{n\to\infty} f_n = 0.$$

# 6 $\mathfrak{L}^p$-Spaces

Given: a measure space $(\Omega, \mathfrak{A}, \mu)$ and $1 \leq p < \infty$. Put $\mathfrak{Z} = \mathfrak{Z}(\Omega, \mathfrak{A})$.

**Definition 1.**

$$\mathfrak{L}^p = \mathfrak{L}^p(\Omega, \mathfrak{A}, \mu) = \left\{ f \in \mathfrak{Z} : \int |f|^p \, d\mu < \infty \right\}.$$

In particular, for $p = 1$: *integrable functions* and $\mathfrak{L} = \mathfrak{L}^1$, and for $p = 2$: *square-integrable functions*. Put

$$\|f\|_p = \left( \int |f|^p \, d\mu \right)^{1/p}, \qquad f \in \mathfrak{L}^p.$$

**Theorem 1 (Hölder inequality).** Let $1 < p, q < \infty$ such that $1/p + 1/q = 1$ and let $f \in \mathfrak{L}^p$, $g \in \mathfrak{L}^q$. Then

$$\int |f \cdot g| \, d\mu \leq \|f\|_p \cdot \|g\|_q.$$

In particular, for $p = q = 2$: *Cauchy-Schwarz inequality.*

*Proof.* See Analysis III or Elstrodt (1996, §VI.1) as well as Theorem 5.3. □

**Theorem 2.** $\mathfrak{L}^p$ is a vector space and $\|\cdot\|_p$ is a semi-norm on $\mathfrak{L}^p$. Furthermore,

$$\|f\|_p = 0 \quad \Leftrightarrow \quad f = 0 \ \mu\text{-a.e.}$$

*Proof.* See Analysis III or Elstrodt (1996, §VI.2). □

**Definition 2.** Let $f, f_n \in \mathfrak{L}^p$ for $n \in \mathbb{N}$. $(f_n)_n$ *converges to $f$ in $\mathfrak{L}^p$ (in mean of order $p$)* if

$$\lim_{n \to \infty} \|f - f_n\|_p = 0.$$

In particular, for $p = 1$: *convergence in mean*, and for $p = 2$: *mean-square convergence*. Notation:

$$f_n \xrightarrow{\mathfrak{L}^p} f.$$

**Remark 1.** Let $f, f_n \in \overline{\mathfrak{Z}}$ for $n \in \mathbb{N}$. Recall (define) that $(f_n)_n$ *converges to $f$ $\mu$-a.e.* if

$$\mu(A^c) = 0$$

for

$$A = \left\{ \lim_{n \to \infty} f_n = f \right\} = \left\{ \limsup_{n \to \infty} f_n = \liminf_{n \to \infty} f_n \right\} \cap \left\{ \limsup_{n \to \infty} f_n = f \right\} \in \mathfrak{A}.$$

Notation:

$$f_n \xrightarrow{\mu\text{-a.e.}} f.$$

**Lemma 1.** Let $f, g, f_n \in \mathfrak{L}^p$ for $n \in \mathbb{N}$ such that $f_n \xrightarrow{\mathfrak{L}^p} f$. Then

$$f_n \xrightarrow{\mathfrak{L}^p} g \quad \Leftrightarrow \quad f = g \ \mu\text{-a.e.}$$

Analogously for convergence almost everywhere.

*Proof.* For convergence in $\mathfrak{L}^p$: '$\Leftarrow$' follows from Theorem 5.4.(ii). Use

$$\|f - g\|_p \leq \|f - f_n\|_p + \|f_n - g\|_p$$

to verify '$\Rightarrow$'.
For convergence almost everywhere: '$\Leftarrow$' trivially holds. Use

$$\left\{\lim_{n\to\infty} f_n = f\right\} \cap \left\{\lim_{n\to\infty} f_n = g\right\} \subset \{f = g\}$$

to verify '$\Rightarrow$'. $\square$

**Theorem 3 (Fischer-Riesz).** Consider a sequence $(f_n)_n$ in $\mathfrak{L}^p$. Then

(i) $(f_n)_n$ Cauchy sequence $\Rightarrow \exists\, f \in \mathfrak{L}^p : \ f_n \xrightarrow{\mathfrak{L}^p} f$ (*completeness*),

(ii) $f_n \xrightarrow{\mathfrak{L}^p} f \Rightarrow \exists$ subsequence $(f_{n_k})_k : f_{n_k} \xrightarrow{\mu\text{-a.e.}} f$.

*Proof.* Ad (i): Consider a Cauchy sequence $(f_n)_n$ and a subsequence $(f_{n_k})_k$ such that

$$\forall\, k \in \mathbb{N}\ \forall\, m \geq n_k : \|f_m - f_{n_k}\|_p \leq 2^{-k}.$$

For

$$g_k = f_{n_{k+1}} - f_{n_k} \in \mathfrak{L}^p$$

we have

$$\Big\| \sum_{\ell=1}^{k} |g_\ell| \Big\|_p \leq \sum_{\ell=1}^{k} \|g_\ell\|_p \leq \sum_{\ell=1}^{k} 2^{-\ell} \leq 1.$$

Put $g = \sum_{\ell=1}^{\infty} |g_\ell| \in \overline{\mathfrak{Z}}_+$. By Theorem 5.1

$$\int g^p \, d\mu = \int \sup_k \Big( \sum_{\ell=1}^{k} |g_\ell| \Big)^p \, d\mu = \sup_k \int \Big( \sum_{\ell=1}^{k} |g_\ell| \Big)^p \, d\mu \leq 1. \tag{1}$$

Thus, in particular, $\sum_{\ell=1}^{\infty} |g_\ell|$ and $\sum_{\ell=1}^{\infty} g_\ell$ converge $\mu$-a.e., see Theorem 5.4.(i). Since

$$f_{n_{k+1}} = \sum_{\ell=1}^{k} g_\ell + f_{n_1},$$

we have

$$f = \lim_{k \to \infty} f_{n_k} \ \mu\text{-a.e.}$$

for some $f \in \mathfrak{Z}$. Furthermore,

$$|f - f_{n_k}| \leq \sum_{\ell=k}^{\infty} |g_\ell| \leq g \ \mu\text{-a.e.},$$

so that, by Theorem 5.5 and (1),

$$\lim_{k \to \infty} \int |f - f_{n_k}|^p \, d\mu = 0.$$

It follows that

$$\lim_{n \to \infty} \|f - f_n\|_p = 0,$$

too. Finally, by Theorem 2, $f \in \mathfrak{L}^p$.
Ad (ii): Assume that

$$f_n \xrightarrow{\mathfrak{L}^p} f.$$

According to the proof of (i) there exists $\widetilde{f} \in \mathfrak{L}^p$ and a subsequence $(f_{n_k})_k$ such that

$$f_{n_k} \xrightarrow{\mu\text{-a.e.}} \widetilde{f} \wedge f_{n_k} \xrightarrow{\mathfrak{L}^p} \widetilde{f}.$$

Use Lemma 1. □

**Example 1.** Let $(\Omega, \mathfrak{A}, \mu) = \big([0,1], \mathfrak{B}([0,1]), \lambda_1|_{\mathfrak{B}([0,1])}\big)$. (By Remark 1.7.(ii) we have $\mathfrak{B}([0,1]) \subset \mathfrak{B}_1$). Define

$$A_1 = [0,1]$$
$$A_2 = [0,1/2], \quad A_3 = [1/2,1]$$
$$A_4 = [0,1/3], \quad A_5 = [1/3,2/3], \quad A_6 = [2/3,1]$$
$$\text{etc.}$$

Put $f_n = 1_{A_n}$. Then

$$\lim_{n\to\infty} \|f_n - 0\|_p = \lim_{n\to\infty} \|f_n\|_p = 0 \tag{2}$$

but

$$\{(f_n)_n \text{ converges}\} = \emptyset.$$

**Remark 2.** Define

$$\mathfrak{L}^\infty = \mathfrak{L}^\infty(\Omega, \mathfrak{A}, P) = \{f \in \mathfrak{Z} : \exists\, c \in \mathbb{R}_+ : |f| \leq c\ \mu\text{-a.e.}\}$$

and

$$\|f\|_\infty = \inf\{c \in \mathbb{R}_+ : |f| \leq c\ \mu\text{-a.e.}\}, \qquad f \in \mathfrak{L}^\infty.$$

$f \in \mathfrak{L}^\infty$ is called *essentially bounded* and $\|f\|_\infty$ is called the *essential supremum of* $|f|$. Use Theorem 4.1.(iii) to verify that

$$|f| \leq \|f\|_\infty\ \mu\text{-a.e.}$$

The definitions and results of this section, except (2), extend to the case $p = \infty$, where $q = 1$ in Theorem 1. In Theorem 3.(ii) we even have $f_n \xrightarrow{\mathfrak{L}^\infty} f \Rightarrow f_n \xrightarrow{\mu\text{-a.e.}} f$.

**Remark 3.** Put

$$\mathfrak{N}^p = \{f \in \mathfrak{L}^p : f = 0\ \mu\text{-a.e.}\}$$

Then the quotient space $L^p = \mathfrak{L}^p / \mathfrak{N}^p$ is a Banach space. In particular, for $p = 2$, $L^2$ is a Hilbert space, with semi-inner product on $\mathfrak{L}^2$ given by

$$\langle f, g \rangle = \int f \cdot g \, d\mu, \qquad f, g \in \mathfrak{L}^2.$$

**Theorem 4.** If $\mu$ is finite and $1 \leq p < q \leq \infty$ then

$$\mathfrak{L}^q \subset \mathfrak{L}^p$$

and

$$\|f\|_p \leq \mu(\Omega)^{1/p - 1/q} \cdot \|f\|_q, \qquad f \in \mathfrak{L}^q.$$

*Proof.* The result trivially holds for $q = \infty$.In the sequel, $q < \infty$. Use $|f|^p \leq 1 + |f|^q$ and Theorem 5.4.(iii) to obtain $\mathfrak{L}^q \subset \mathfrak{L}^p$. Put $r = q/p$ and define $s$ by $1/r + 1/s = 1$. Theorem 1 yields

$$\int |f|^p \, d\mu \leq \left( \int |f|^{p\cdot r} \, d\mu \right)^{1/r} \cdot \big(\mu(\Omega)\big)^{1/s}.$$

□

**Example 2.** Let1 $\leq p < q \leq \infty$. With respect to the counting measure on $\mathfrak{P}(\mathbb{N})$, $\mathfrak{L}^p \subset \mathfrak{L}^q$. With respect to the Lebesgue measure on $\mathfrak{B}_k$ neither $\mathfrak{L}^q \subset \mathfrak{L}^p$ nor $\mathfrak{L}^p \subset \mathfrak{L}^q$.

# 7 The Radon-Nikodym-Theorem

Given: a measure space $(\Omega, \mathfrak{A}, \mu)$. Put $\overline{\mathfrak{Z}}_+ = \overline{\mathfrak{Z}}_+(\Omega, \mathfrak{A})$.

**Definition 1.** For $f$ (quasi-)$\mu$-integrable and $A \in \mathfrak{A}$, the *integral of $f$ over $A$* is

$$\int_A f \, d\mu = \int 1_A \cdot f \, d\mu.$$

(Note: $|1_A \cdot f| \leq |f|$.)

**Theorem 1.** Let $f \in \overline{\mathfrak{Z}}_+$ and put

$$\nu(A) = \int_A f \, d\mu, \qquad A \in \mathfrak{A}.$$

Then $\nu$ is a measure on $\mathfrak{A}$.

*Proof.* Clearly $\nu(\emptyset) = 0$ and $\nu \geq 0$. For $A_1, A_2, \ldots \in \mathfrak{A}$ pairwise disjoint

$$\begin{aligned} \nu\Big(\bigcup_{i=1}^{\infty} A_i\Big) &= \int \sum_{i=1}^{\infty} 1_{A_i} \cdot f \, d\mu = \int \lim_{n\to\infty} \Big(\sum_{i=1}^{n} 1_{A_i} \cdot f\Big) \, d\mu \\ &= \lim_{n\to\infty} \int \sum_{i=1}^{n} 1_{A_i} \cdot f \, d\mu = \sum_{i=1}^{\infty} \int 1_{A_i} \cdot f \, d\mu \\ &= \sum_{i=1}^{\infty} \nu(A_i) \end{aligned}$$

follows from Theorem 5.1. □

**Definition 2.** The mapping $\nu$ in Theorem 1 is called *measure with $\mu$-density $f$*, or distribution with density $f$. Notation: $\nu = f \cdot \mu$ (bad, but common notation: $\mathrm{d}\nu = d \cdot \mathrm{d}\mu$). If $\int f \, d\mu = 1$ then $f$ is called *probability density*.

**Example 1.** The introductory examples of probability spaces were defined by means of probability densities.

(i) Let $(\Omega, \mathfrak{A}, \mu) = (\mathbb{R}^k, \mathfrak{B}_k, \lambda_k)$. For

$$f(x) = (2\pi)^{-k/2} \cdot \exp\left(-\tfrac{1}{2} \textstyle\sum_{i=1}^{k} x_i^2\right)$$

we get the *$k$-dimensional standard normal distribution $\nu$*.

For $B \in \mathfrak{B}_k$ such that $0 < \lambda_k(B) < \infty$ and

$$f = \frac{1}{\lambda_k(B)} \cdot 1_B$$

we get the *uniform distribution on $B$*.

(ii) $\Omega = \mathbb{N}$, $\mathfrak{A} = \mathfrak{P}(\mathbb{N})$, $\mu$ the counting measure. A mapping $f : \Omega \to \mathfrak{R}_+$ (i.e., a sequence) is in $\mathfrak{L}^1$ iff it is an absolutely summable sequence (see Übung4.3a)), and for each such $f$ and $A \subseteq \Omega$,

$$\forall A \in \mathfrak{A} : \nu(A) = \int_A f \, d\mu = \sum_{n \in A} f(n). \tag{1}$$

Conversely, *any* measure $\nu$ on $\mathfrak{A}$ is a measure with density with respect to $\mu$: Put $f(\omega) := \nu(\{\omega\})$, then ((1)) holds.

**Theorem 2.** Let $\nu = f \cdot \mu$ with $f \in \overline{\mathfrak{Z}}_+$ and $g \in \overline{\mathfrak{Z}}(\Omega, \mathfrak{A})$. Then

$$g \text{ (quasi)-}\nu\text{-integrable} \quad \Leftrightarrow \quad g \cdot f \text{ (quasi)-}\mu\text{-integrable},$$

in which case

$$\int g \, d\nu = \int g \cdot f \, d\mu$$

*Proof.* First, assume that $g = 1_A$ with $A \in \mathfrak{A}$. Then the statements hold by definition. For $g \in \Sigma_+(\Omega, \mathfrak{A})$ we now use linearity of the integral. For $g \in \overline{\mathfrak{Z}}_+$ we take a sequence $(g_n)_{n \in \mathbb{N}}$ in $\Sigma_+(\Omega, \mathfrak{A})$ such that such that $g_n \uparrow g$. Then $g_n \cdot f \in \overline{\mathfrak{Z}}_+$ and $g_n \cdot f \uparrow g \cdot f$. Hence, by Theorem 5.1 and the previous part of the proof

$$\int g \, d\nu = \lim_{n \to \infty} \int g_n \, d\nu = \lim_{n \to \infty} \int g_n \cdot f \, d\mu = \int g \cdot f \, d\mu.$$

Finally, for $g \in \overline{\mathfrak{Z}}(\Omega, \mathfrak{A})$ we already know that

$$\int g^\pm d\nu = \int g^\pm \cdot f \, d\mu = \int (g \cdot f)^\pm \, d\mu.$$

Use linearity of the integral. □

**Remark 1.**

$$f, g \in \overline{\mathfrak{Z}}_+ \wedge f = g \ \mu\text{-a.e.} \quad \Rightarrow \quad f \cdot \mu = g \cdot \mu.$$

**Theorem 3 (Uniqueness of densities).** Let $f, g \in \overline{\mathfrak{Z}}_+$ such that $f \cdot \mu = g \cdot \mu$. Then

(i) $f$ $\mu$-integrable $\Rightarrow f = g$ $\mu$-a.e.,

(ii) $\mu$ $\sigma$-finite $\Rightarrow f = g$ $\mu$-a.e.

*Proof.* Ad (i): It suffices to verify the claim: If $f, g$ $\mu$-integrable and

$$\forall A \in \mathfrak{A} : \int_A f \, d\mu \leq \int_A g \, d\mu \quad \Rightarrow \quad f \leq g \ \mu\text{-a.e.}$$

To this end, take $A = \{f > g\}$. By assumption,

$$-\infty < \int_A f \, d\mu \leq \int_A g \, d\mu < \infty$$

and therefore $\int_A (f-g)\,d\mu \leq 0$. However,

$$1_A \cdot (f-g) \geq 0,$$

hence $\int_A (f-g)\,d\mu \geq 0$. Thus

$$\int 1_A \cdot (f-g)\,d\mu = 0.$$

Theorem 5.3 implies $1_A \cdot (f-g) = 0$ $\mu$-a.e., and by definition of $A$ we get $\mu(A) = 0$.
Ad (ii): Assume first that $\mu$ is finite. Since for all $k \in \mathbb{N}$,

$$\infty \cdot \mu(\{f=\infty\} \setminus \{g \geq k\}) = \int_{\{f=\infty\}\setminus\{g\geq k\}} f \mathrm{d}\mu = \int_{\{f=\infty\}\setminus\{g\geq k\}} g \mathrm{d}\mu \leq k\mu(\Omega)\ ,$$

we have that $\mu(\{f=\infty\} \setminus \{g \geq k\}) = 0$, and by $\sigma$–continuity from below, $\mu(\{f = \infty \setminus \{g=\infty\}\}) = 0$. By symmetry, we conclude

$$\mu(\{f=\infty\}\Delta\{g=\infty\}) = 0\ .$$

Set $A_0 = \{f=\infty\} \cup \{g=\infty\}$, $A_1 = A_0^c$; then $\mathbf{1}_{A_0} f = \mathbf{1}_{A_0} g$ $\mu$–a.e., and we claim that

$$\mathbf{1}_{A_1} f = \mathbf{1}_{A_1} g \quad \mu\text{–a.e.}\ . \tag{2}$$

Since

$$A_1 \cap \{f > g\} = \bigcap_{n\in\mathbb{N}} \underbrace{\{n > f > g + 1/n\}}_{=:C_n}\ ,$$

we just have to show $\mu(C_n) = 0$. But

$$\int \mathbf{1}_{C_n} g \mathrm{d}\mu = \int \mathbf{1}_{C_n} f \mathrm{d}\mu \geq \int \mathbf{1}_{C_n}(g + 1/n) = \int \mathbf{1}_{C_n} g \mathrm{d}\mu + \mu(B_n)/n\ .$$

Since further

$$\int \mathbf{1}_{C_n} g \mathrm{d}\mu = \int \mathbf{1}_{C_n} f \mathrm{d}\mu \leq n \cdot \mu(\Omega) < \infty\ ,$$

this entails $\mu(C_n) = 0$, and hence $\mu(A_1 \cap \{f > g\}) = 0$; by symmetry, also $\mu(A_1 \cap \{g > f\}) = 0$, i.e., (2) follows.
Let now $\mu$ be just $\sigma$–finite, and let $B_n \in \mathfrak{A}$ be disjoint such that $\mu(B_n) < \infty$, $\bigcup_n B_n = \Omega$. Set $\mu_n(A) := \mu(A \cap B_n)$. Then $\mu_n$ are measures, and for all $A \in \mathfrak{A}$,

$$\mu(A) = \sum_n \mu_n(A)\ .$$

Moreover, $f \cdot \mu_n = g \cdot \mu_n$, and by the first part we know that

$$f = g \qquad \mu_n - -\text{a.e.}, \quad \forall n \in \mathbb{N}\ .$$

But then

$$\mu(\{f\ not = g\}) = \sum_n \mu_n(\{f \neq g\}) = 0\ .$$

□

**Remark 2.** Let $(\Omega, \mathfrak{A}, \mu) = (\mathbb{R}^k, \mathfrak{B}_k, \lambda_k)$ and $x \in \mathbb{R}^k$. There is no density $f \in \overline{\mathfrak{Z}}_+$ w.r.t. $\lambda_k$ such that $\delta_x = f \cdot \lambda_k$ (recall $\delta_x$ the Dirac point measure). This follows from $\varepsilon_x(\{x\}) = 1$ and

$$(f \cdot \lambda_k)(\{x\}) = \int_{\{x\}} f \, d\lambda_k = 0.$$

**Definition 3.** A measure $\nu$ on $\mathfrak{A}$ is *absolutely continuous w.r.t.* $\mu$ if

$$\forall\, A \in \mathfrak{A} : \mu(A) = 0 \Rightarrow \nu(A) = 0.$$

Notation: $\nu \ll \mu$.

**Remark 3.**

(i) $\nu = f \cdot \mu \Rightarrow \nu \ll \mu$.

(ii) In Remark 2 neither $\varepsilon_x \ll \lambda_1$ nor $\lambda_1 \ll \varepsilon_x$.

(iii) Let $\mu$ denote the counting measure on $\mathfrak{A}$. Then $\nu \ll \mu$ for every measure $\nu$ on $\mathfrak{A}$.

(iv) Let $\mu$ denote the counting measure on $\mathfrak{B}_1$. Then there is no density $f \in \overline{\mathfrak{Z}}_+$ such that $\lambda_1 = f \cdot \mu$.

**Lemma 1.** Let $f_n \xrightarrow{\mathfrak{L}^p} f$ and $A \in \mathfrak{A}$. If $p = 1$ or $\mu(A) < \infty$ then

$$\int_A f_n \, d\mu \to \int_A f \, d\mu.$$

*Proof.* For $p = 1$, this follows from

$$\Big| \int_A f_n \, d\mu - \int_A f \, d\mu \Big| \le \int_A |f_n - f| \, \mathrm{d}\mu \to 0 \; ;$$

if $\mu(A) < \infty$ and $p > 1$ set $1/q = 1 - 1/p$; then by Theorem 6.1,

$$\int \mathbf{1}_A \cdot |f_n - f| \, \mathrm{d}\mu \le \underbrace{\Big( \int \mathbf{1}_A^q \Big)^{1/q}}_{= \mu(A)^{1/q} < \infty} \cdot \underbrace{\Big( \int |f - f_n|^p \Big)^{1/p}}_{\to 0} .$$

□

**Theorem 4 (Radon, Nikodym).** For every $\sigma$-finite measure $\mu$ and every measure $\nu$ on $\mathfrak{A}$ we have

$$\nu \ll \mu \quad \Rightarrow \quad \exists\, f \in \overline{\mathfrak{Z}}_+ : \nu = f \cdot \mu.$$

*Proof.* We will prove this only for finite measures (since we need it only for finite measures).

**Step 1:** We assume the stronger condition

$$\forall\, A \in \mathfrak{A} : \nu(A) \le \mu(A) \wedge \mu(\Omega) < \infty.$$

A class $\mathfrak{U} = \{A_1, \dots, A_n\}$ is called a (finite measurable) partition of $\Omega$ iff $A_1, \dots, A_n \in \mathfrak{A}$ are pairwise disjoint and $\bigcup_{i=1}^n A_i = \Omega$. The set of all partitions is partially ordered by

$$\mathfrak{U} \sqsubset \mathfrak{V} \quad \text{iff} \quad \forall\, A \in \mathfrak{U}\ \exists\, B \in \mathfrak{V} : A \subset B.$$

The infimum of two partitions is given by

$$\mathfrak{U} \wedge \mathfrak{V} = \{A \cap B : A \in \mathfrak{U}, B \in \mathfrak{V}\}\,.$$

For any partition $\mathfrak{U}$ we define

$$f_{\mathfrak{U}} = \sum_{A \in \mathfrak{U}} \alpha_A \cdot 1_A$$

with

$$\alpha_A = \begin{cases} \nu(A)/\mu(A) & \text{if } \mu(A) > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Clearly $f_{\mathfrak{U}} \in \Sigma_+(\Omega, \sigma(\mathfrak{U})) \subset \Sigma_+(\Omega, \mathfrak{A})$, $\sigma(\mathfrak{U}) = \mathfrak{U}^+ \cup \{\emptyset\}$, and

$$\forall\, A \in \sigma(\mathfrak{U}) : \nu(A) = \int_A f_{\mathfrak{U}}\, d\mu.$$

(Thus we have $\nu|_{\sigma(\mathfrak{U})} = f_{\mathfrak{U}} \cdot \mu|_{\sigma(\mathfrak{U})}$.) Let $\mathfrak{U} \sqsubset \mathfrak{V}$ and $A \in \mathfrak{V}$. Then

$$\nu(A) = \int_A f_{\mathfrak{V}}\, d\mu = \int_A f_{\mathfrak{U}}\, d\mu,$$

since $A \in \sigma(\mathfrak{U})$. Hence

$$\int_A f_{\mathfrak{V}}^2\, d\mu = \int_A f_{\mathfrak{V}} \cdot f_{\mathfrak{U}}\, d\mu,$$

since $f_{\mathfrak{V}}|_A$ is constant, and therefore

$$0 \le \int (f_{\mathfrak{U}} - f_{\mathfrak{V}})^2\, d\mu = \int f_{\mathfrak{U}}^2\, d\mu - \int f_{\mathfrak{V}}^2\, d\mu. \tag{3}$$

Put

$$\beta = \sup\left\{ \int f_{\mathfrak{U}}^2\, d\mu : \mathfrak{U} \text{ partition} \right\},$$

and note that $0 \le \beta \le \mu(\Omega) < \infty$, since $f_{\mathfrak{U}} \le 1$. Consider a sequence of functions $f_n = f_{\mathfrak{U}_n}$ such that

$$\lim_{n \to \infty} \int f_n^2\, d\mu = \beta.$$

Due to (3) we may assume that $\mathfrak{U}_{n+1} \sqsubset \mathfrak{U}_n$. Then, by (3), $(f_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in $\mathfrak{L}^2$, so that there exists $f \in \mathfrak{L}^2$ with

$$\lim_{n \to \infty} \|f_n - f\|_2 = 0 \quad \wedge \quad 0 \le f \le 1\ \mu\text{-a.e.},$$

see Theorem 6.3.

We claim that $\nu = f \cdot \mu$. Let $A \in \mathfrak{A}$. Put

$$\widetilde{\mathfrak{U}}_n = \mathfrak{U}_n \wedge \{A, A^c\}$$

and

$$\widetilde{f}_n = f_{\widetilde{\mathfrak{U}}_n}.$$

Then

$$\nu(A) = \int_A \widetilde{f}_n \, d\mu = \int_A f_n \, d\mu + \int_A (\widetilde{f}_n - f_n) \, d\mu,$$

and (3) yields $\lim_{n\to\infty} \|\widetilde{f}_n - f_n\|_2 = 0$. It remains to apply Lemma 1.
**Step 2:** We assume only that $\mu, \nu$ are finite, and $\nu \ll \nu$. Then $\mu, \nu \leq \mu + \nu =: \tau$; by Step 1, we have densities $g, h : \Omega \to [0,1]$ with $\mu = g \cdot \tau$, $\nu = h \cdot \tau$. Since

$$\mu(\{g = 0\}) = \int_{\{g=0\}} \mathrm{d}\mu = \int_{\{g=0\}} g \; \mathrm{d}\tau = 0$$

and $\nu \ll \mu$, $\nu(\{g = 0\}) = 0$. The function

$$f(x) := \begin{cases} h(x)/g(x), & g(x) \neq 0, \\ 0, & g(x) = 0, \end{cases}$$

is now a density for $\nu$:

$$\nu(A) = \int_{A\cap\{g\neq 0\}} \underbrace{h}_{=fg} \; \mathrm{d}\tau = \int_{A\cap\{g\neq 0\}} f \; \mathrm{d}\mu = \int_A f \; \mathrm{d}\mu \; .$$

□