Probability Theory

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Skript nach einer Vorlage von
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Darmstadt, WS 2006/07

Literature

In particular,

H. Bauer, *Probability Theory*, de Gruyter, Berlin, 1996.

P. Billingsley, *Probability and Measure*, Wiley, New York, first edition, 1979, third edition, 1995.

J. Elstrodt, *Maß- und Integrationstheorie*, Springer, Berlin, first edition, 1996, third edition, 2002.

P. Gänssler, W. Stute, *Wahrscheinlichkeitstheorie*, Springer, Berlin, 1977.

Contents

Chapter I

Introduction

A *stochastic model*: a probability space $(\Omega, \mathfrak{A}, P)$ together with a collection of random variables (measurable mappings) $\Omega \to \mathbb{R}$, say.

Examples of probability spaces, known from 'Introduction to Statistics' or 'Analysis':

(i) Given: a countable set Ω and $f : \Omega \to \mathbb{R}_+$ such that $\sum_{\omega \in \Omega} f(\omega) = 1$.
Take the power set $\mathfrak{A} = \mathfrak{P}(\Omega)$ and define

$$P(A) = \sum_{\omega \in A} f(\omega), \qquad A \subset \Omega.$$

(ii) Given: $f : \mathbb{R}^k \to \mathbb{R}_+$ such that $\int_{\mathbb{R}^k} f(\omega)\, d\omega = 1$.
Let $\Omega = \mathbb{R}^k$, take the σ-algebra $\mathfrak{A} = \mathfrak{B}(\mathbb{R}^k)$ of Borel sets in $\mathbb{R}^k$ and define

$$P(A) = \int_A f(\omega)\, d\omega, \qquad A \in \mathfrak{B}(\mathbb{R}^k).$$

Main topics in this course:

(i) construction of probability spaces, including the theory of measure and integration,

(ii) limit theorems,

(iii) conditional probabilities and expectations.

Example 1. Limit theorems like the law of large numbers or the central limit theorem deal with sequences $X_1, X_2, \dots$ of random variables and their partial sums

$$S_n = \sum_{i=1}^{n} X_i$$

(gambling: cumulative gain after n trials; physics: position of a particle after n collisions).

Under which conditions and in which sense does S_n/n or $S_n/\sqrt{n}$ converge, as n tends to infinity?

Example 2. Limit theorems hold in particular for independent and identically distributed (i.i.d.) random variables $X_1, X_2, \ldots$ with $\mathrm{E}(X_i) = 0$ and $\mathrm{Var}(X_i) = 1$. Then S_n/n 'converges' to zero and $S_n/\sqrt{n}$ 'converges' to the standard normal distribution.

In particular, in a simple case of gambling: X_i takes values ± 1 with probability $1/2$. Existence of such a model? Existence for every choice of the distribution of X_i?

Example 3. The fluctuation of a stock price defines a function on the time interval $\mathbb{R}_+$ with values in $\mathbb{R}$ (for simplicity, we admit negative stock prices at this point). What is a reasonable σ-algebra on the space Ω of all mappings $\mathbb{R}_+ \to \mathbb{R}$ or on the subspace of all continuous mappings? How can we define (non-discrete) probability measures on these spaces in order to model the random dynamics of stock prices? Analogously for random perturbations in physics, biology, etc.

More generally, the same questions arise for mappings $I \to S$ with an arbitrary non-empty set I and $S \subset \mathbb{R}^d$ (physics: phase transition in ferromagnetic materials, the orientation of magnetic dipoles on a set I of sites; medicine: spread of diseases, certain biometric parameters for a set I of individuals; environmental science: the concentration of certain pollutants in a region I).

Example 4. Consider two random variables X_1 and X_2. If $P(\{X_2 = v\}) > 0$ then the conditional probability of $\{X_1 \in A\}$ given $\{X_2 = v\}$ is defined by

$$P(\{X_1 \in A\} \,|\, \{X_2 = v\}) = \frac{P(\{X_1 \in A\} \cap \{X_2 = v\})}{P(\{X_2 = v\})}.$$

How can we reasonably extend this definition to the case $P(\{X_2 = v\}) = 0$, e.g., for X_2 being normally distributed? How does the observation $X_2 = v$ change our stochastic model? Cf. Example 3.