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14. Aufgabenblatt zur Vorlesung
"'Probability Theory"'

1. – warming up

(1) Compute $\mathbb{E}(X \mid Y)$ in the following concrete situations:

(a) Let $Z = (Z_1, Z_2)$ be equidistributed on $\{0,1\}^2$ (two coins tossed), $X = Z_1 + Z_2$ and $Y = Z_1$.

(b) Let X be exponential with parameter 1, and let Y be 0 if $X \leq 1$ and $Y = X$ if $X > 1$. (We measure X; below some threshhold, we can't measure.)

(2) Let X, Y be random variables.

(a) Compute $\mathbb{E}(\alpha X + \beta Y \mid Y)$. Apply this to $X = \sum_{1<i\leq n} X_i$, $Y = X_1$ where $(X_i)_{i\in\in\mathbb{N}}$ is iid.

(b) Give an example where $\mathbb{E}(X \mid X+Y) \neq \mathbb{E}(X \mid X) + \mathbb{E}(X \mid Y)$. (Choose $X = -Y$ and use Remark V.1.1.1 to compute $\mathbb{E}(X \mid Y)$.)

(c) Let now X, Y be iid with mean m. Is the following correct?

$$\mathbb{E}(X \mid X+Y=z) = \mathbb{E}(X \mid X = z - Y) = \mathbb{E}(z - Y) = z - m\ .$$

(d) Let X, Y be independent and $\mathbb{P}(Y=1) = 1/2 = \mathbb{P}(Y=-1)$. Compute $\mathbb{E}(YX \mid X)$. Apply this to the case $X = |g|$, $Y = \text{sign}(g)$ where g is a $\mathcal{N}(0,1)$–distributed random variable[1] to compute $\mathbb{E}(g \mid |g|)$.

Solution:

2. Assume that the distribution (X, Y) has a Lebesgue density $p(x,y)$.

(a) Prove that the distributions of X, Y have the Lebesgue densities

$$p_X(x) = \int_{\mathbb{R}} p(x,y)dy \qquad p_Y(y) = \int_{\mathbb{R}} p(x,y)dx\ .$$

(b) Derive a formula for the conditional expectation $\mathbb{E}(X \mid Y = y)$ using p, p_X, p_Y.

3. Prove or disprove: For any $X, \mathfrak{G}$ we have

$$\{X > c\} \subseteq \{\mathbb{E}(X \mid \mathfrak{G}) > c\}\ .$$

4. The conditional variance is defined as

$$\text{var}(X \mid Y) := \mathbb{E}\left[(X - \mathbb{E}(X \mid Y))^2 \mid Y\right]\ .$$

Prove (and interprete) the law of total variance,

$$\text{var}(X) = \mathbb{E}[\text{var}(X \mid Y)] + \text{var}[\mathbb{E}(X \mid Y)]\ .$$

[1] Take for granted that X and Y are indeed independent