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11. Aufgabenblatt zur Vorlesung
"'Probability Theory"'

1. – warming up Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of r.v. ,and let $\mathfrak{A}_n = \sigma(X_n)$. Which of the following events are terminal events? Which can you write with the help of sets of the form $\overline{\lim}$, $\underline{\lim}$?

1. $\{X_n$ is monotonely increasing$\}$;
2. $\{X_n \le 1/n \ \forall n \in \mathbb{N}\}$;
3. $\{\overline{\lim}_n nX_n > 1\}$;
4. $\{X_n \to 0\}$;
5. $\{X_n \ge \max\{X_1, \dots, X_n\}$ infinitely often$\}$;
6. $\{X_{2n} \ge \max\{X_n, \dots, X_{2n}\}$ infinitely often$\}$.

2.

1. Assume that $A_n \uparrow A$ and show that $\underline{\lim}_n A_n = \overline{\lim}_n A_n = A$.
2. Let us write $A_n \to A$ iff $\underline{\lim}_n A_n = A = \overline{\lim}_n A_n$. Formulate and prove a sandwich lemma.
3. Verify the facts that are stated in Remark IV.1.2..
4. Prove or disprove:

$$\underline{\lim}_n (A_n \cap B_n) = (\underline{\lim}_n A_n) \cap (\underline{\lim}_n B_n), \qquad \underline{\lim}_n (A_n \cup B_n) = (\underline{\lim}_n A_n) \cup (\underline{\lim}_n B_n) .$$

3. Let $(X_n)_{n\in\mathbb{N}}$ be i.i.d. with $X_1 \in \mathfrak{L}^2$. Put $S_n = \sum_{i=1}^n X_i$.

1. Show that
$$\frac{1}{n-1} \cdot \sum_{i=1}^n (X_i - S_n/n)^2 \overset{P\text{-a.s.}}{\longrightarrow} \operatorname{Var}(X_1),$$
i.e., the sample variance converges almost surely to the population variance. (Use the Strong Law of Large Numbers.)
2. Assume that $\mathbb{E}\, X_1 = 0$, and set $a_n = n(\log n)^{1/2+\delta}$. Prove that a.s. we have $\overline{\lim}_n |X_n|/a_n = 0$. (Use Borell–Cantelli and Chebyshev's inequality to show that a.s., $\{|X_n| \ge \varepsilon a_n\}$ does not happen infinitely often.)

4. The SLLN yields an idea called the *Monte Carlo Method of direct simulation.* Let $a \in \mathbb{R}$ be an unknown number; if one can construct a r.v. X such that $\mathbb{E}\,X = a$, then n independent simulations $X_1, \ldots, X_n$ of X will satisfy $\frac{1}{n} S_n := \frac{1}{n} \sum_{i \le n} X_i \to a$ almost surely. (Interestingly, it is often much easier to find and simulate such an X than to compute a directly.)

1. Calculate the variance of the error of this method, $\mathbb{E}\,(S_n/n - a)^2$.

2. As a special case, consider a function $f : [0,1]^d \to \mathbb{R}$ and set $a = \int_{[0,1]^d} f(x)dx$. Let U be uniformly distributed on $[0,1]^d$. Prove that $X := f(U)$ satisfies

$$\mathbb{E}\,X = a, \qquad \mathrm{Var}\,(X) \le \int_{[0,1]^d} f^2(x)dx\ ,$$

   and study the corresponding direct simulation method.

3. Perform numerical experiments for the method studied above. (Use uniformly distributed random numbers from $[0,1]$ that are available on your computer.) Consider, in particular, test functions

$$f(x) = \exp\Big(-\sum_{j=1}^{d} c_j \cdot |x_j - w_j|\Big)$$

   with constants $c_j > 0$ and $0 < w_j < 1$