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WS 2006/07
15.12.06

9. Aufgabenblatt zur Vorlesung "'Probability Theory"'

1. – warming up

(a) Let $X_n \sim \Gamma(\alpha_n, \beta_n)$, i.e., P_{X_n} has Lebesgue density

$$g(x; \alpha_n, \beta_n) := \mathbb{1}_{x>0} x^{\alpha_n - 1} \cdot \frac{\beta_n^{\alpha_n} e^{-\beta_n}}{\Gamma(\alpha_n)} ,$$

where Γ denotes the Gamma function, and $\alpha_n, \beta_n > 0$. Assume that $\alpha_n \to \alpha$, $\beta_n \to \beta$ with $\alpha, \beta \in [0, \infty]$. When does X_n converge in distribution?

(b) Consider $\mathbb{N}$ with metric $d(x, y) := |x - y|$. Show that in this metric space, every set $\{x\}$ and thus any subset $A \subseteq \mathbb{N}$ is open; conclude that $\mathfrak{B}(\mathbb{N}) = \mathfrak{P}(\Omega)$. Prove that for probability measures Q, Q_n on $\mathfrak{P}(\Omega)$, we have

$$Q_n \xrightarrow{w} Q \quad \Leftrightarrow \quad \forall_{k \in \mathbb{N}} \; Q_n(\{k\}) \to Q(\{k\}) .$$

(c) Let $p_n \in (0, 1)$ such that $n \cdot p_n \to \lambda > 0$. Show that $B(n, p_n) \xrightarrow{w} Poi(\lambda)$. (*Hint:* Use (b) and convergence rules from Analysis. In particular, recall that, if $a_n \to a > 0$, then $(1 - a_n/n)^n \to e^{-a}$.)

2. Let F_n, F be distribution functions. Assume that F is continuous and that $F_n(x) \to F(x)$ for all $x \in \mathbb{R}$. Prove that

$$\sup_{x \in \mathbb{R}} |F_n(x) - F(x)| \to 0 .$$

(*Hint:* Show first that for fixed N, $\sup_{x \in [-N, N]} |F_n(x) - F(x)| \to 0$; for this, use the uniform continuity of F on $[-N, N]$.)

3. Let $\Omega =]0, 1[$, $\mathfrak{A} = \mathfrak{B}(\Omega)$, and consider the uniform distribution P on Ω. For a distribution function F, define

$$X(\omega) = \inf\{z \in \mathbb{R} : \omega \leq F(z)\}, \qquad \omega \in]0, 1[.$$

(a) Show that X is a random variable with $F_X = F$.

(b) Let U be uniformly distributed on $]0, 1[$. Determine a measurable mapping $T :]0, 1[\to [0, \infty[$ such that $T(U)$ is exponentially distributed with parameter $\lambda > 0$. (This is a state-of-the-art way to simulate exponential distributions.)

4. Consider the set of *molecular* probability measures on $\mathbb{R}$, i.e.,

$$\mathfrak{P} = \left\{ \sum_{k=1}^{n} \lambda_k \cdot \delta_{x_k} : n \in \mathbb{N}, \; \lambda_k > 0, \; \sum_{k=1}^{n} \lambda_k = 1, \; x_k \in \mathbb{R} \right\} .$$

Prove that $\mathfrak{P}$ is dense in the set of all probability measures on $(\mathbb{R}, \mathfrak{B})$ w.r.t. weak convergence, i.e., for every probability measure μ on $(\mathbb{R}, \mathfrak{B})$ a suitable sequence in $\mathfrak{P}$ converges weakly to μ.