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8. Aufgabenblatt zur Vorlesung "'Probability Theory"'

1. (Warming up)

(a) Let $(\Omega, \mathfrak{A}, P) = ([0,1], \mathfrak{B}, \lambda_1)$, let $x_n, x \in \mathbb{R}$, set $X_n(\omega) := x_n$, $X(\omega) := x$ for all $\omega \in \mathbb{R}$. When does convergence in probability/a.s. hold?

(b) Let $\lambda_n > 0$, and let $Y_n \sim Exp(\lambda_n)$. Assume that $\lambda_n \to \lambda \in [0, \infty]$. Does the sequence Y_n converge in distribution?

(c) Consider the product space $\Omega = \bigotimes_{i \in \mathbb{N}} \mathbb{R}, \mathfrak{A} = \bigotimes_{i \in \mathbb{N}} \mathfrak{B}, P = \bigotimes_{i \in \mathbb{N}} \mu$, where μ is the $Exp(1)$ – distribution. Show that the series $Y_n(\omega) := \min\{\omega, \ldots, \omega_n\}$ converges in probability to zero. Do you think that the series $Z_n(\omega) := \omega_n$ should converge in probability?

2.

(a) Let $(\Omega, \mathfrak{A}, P)$ be a countable probability space. Prove that then for $X_n, X : \Omega \to \mathbb{R}$ with $X_n \xrightarrow{P} X$ we even have $X_n \to X$ P–a.s..

(b) Let now $(\Omega, \mathfrak{A}, P)$ be arbitrary, and assume that $X_n \xrightarrow{P} X$; further, assume that $X_n(\omega) \leq X_{n+1}(\omega)$ for all n, ω. Does it hold that $X_n \to X$ P–a.s.?

3. Let f_n and f denote probability densities w.r.t. the Lebesgue measure.

(a) Suppose that f_n converges to f λ_1-almost everywhere. Show that $f_n \cdot \lambda_1$ converges weakly to $f \cdot \lambda_1$. (*Hint*: Portmanteau theorem.)

(b) (*) Provide an example, where $f_n \cdot \lambda_1$ converges weakly to $f \cdot \lambda_1$ but f_n converges to f only on a set of Lebesgue measure zero.

4.

(a) (*) Let $Q_n = N(\mu_n, \sigma_n^2)$, where $\mu_n \in \mathbb{R}$ and $\sigma_n \geq 0$. Show that $(Q_n)_{n \in \mathbb{N}}$ converges weakly iff $(\mu_n)_{n \in \mathbb{N}}$ and $(\sigma_n)_{n \in \mathbb{N}}$ converge. (Compare with Ex.III.3.1(b)).

(b) Let X_n be geometrically distributed with parameter $p_n = \lambda/n$. Find the limes of X_n/n in distribution. (*Hint:* Calculate $\log P(X_n/n > t)$, then use Theorem III.3.2.)