TU Darmstadt
Fachbereich Mathematik
Jakob Creutzig

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7. Aufgabenblatt zur Vorlesung
"‘Probability Theory"’

1. – **Warming up** Let X, Y be random variables.

(a) Prove or disprove: $E(X+Y) = EX + EY$.

(b) Prove or disprove: $E(X \cdot Y) = (EX) \cdot (EY)$.

(c) Prove or disprove: $Var(X+Y) = Var(X) + Var(Y)$.

(d) Reformulate Theorem II.2.1, II.2.5, II.5.1 and II.5.5 in the language of random variables and expectations.

2. For $i \in \{1, \dots, n\}$, let $(\Omega_i, \mathfrak{A}_i) = (\mathbb{R}, \mathfrak{B})$ with the measures $\mu_i = f_i \cdot \lambda$, where

$$f_i(t) = \mathbb{1}_{[0,\infty[}(t) \cdot \alpha_i \cdot e^{-\alpha_i t}$$

for some constant $\alpha_i > 0$.

(a) Show that μ_i is a probability measure.

(b) Consider, on $\Omega = \bigotimes_{i \le n} \Omega_i$, the product measure $\bigotimes_{i \le n} \mu_i$, and set

$$X_i : \Omega \to \mathbb{R}, \qquad X_i((\omega_1, \dots, \omega_n)) := \omega_i \ .$$

Prove that X_i are random variables, and determine their distribution.

(c) Prove that the mapping $Y(\omega) := \min\{X_1(\omega), \dots, X_n(\omega)\}$ is also a random variable, and determine its distribution.

3. Let $\Omega_i = \{0,1\}$, $\mathfrak{A}_i = \mathfrak{P}(\Omega_i)$, and $\mu_i = p \cdot \delta_1 + (1-p) \cdot \delta_0$ for $p \in]0,1[$ and $i \in \mathbb{N}$. Consider the corresponding product space $(\Omega, \mathfrak{A}, P)$.

a) Determine, for $1 \le n < m$, the distribution of the random variable

$$X_{n,m} : \Omega \to \{0, \dots, m-n\} : X_n(\omega) = \omega_n + \cdots + \omega_m.$$

What is the ditribution of $X_{n,m} + X_{k,l}$ if $n < m < k < l$?

b) Construct a random variable on $(\Omega, \mathfrak{A}, P)$ that is geometrically distributed with parameter p.

c) Construct random variables X and Y on $(\Omega, \mathfrak{A}, P)$ that do not coincide almost surely but have the same distribution.

4. Let $(\Omega_i, \mathfrak{A}_i) = (\{0,1\}, \mathfrak{P}(\{0,1\}))$ and μ_i the uniform distribution on Ω_i. Consider the product space $\Omega = \bigotimes_{i \in \mathbb{N}} \Omega_i$ with the product measure $P = \bigotimes_{i \in \mathbb{N}} \mu_i$.

(a) Prove that *any* mapping $Y : \Omega \to \mathbb{R}$ which depends solely on the first n coordinates is measurable.

(b) Show that the mapping

$$X : \Omega \to [0,1], \qquad X(\omega) := \sum_{i=1}^{\infty} 2^{-i} \cdot \omega_i$$

is a random variable. (Write X as limit of random variables.)

(c)(*) Determine P_X. (Evaluate P_X on intervals of the form $[i/2^k, (i+1)/2^k[$.)