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6. Aufgabenblatt zur Vorlesung
"'Probability Theory"'

1. – **Warming up** Pick three of the following problems, and construct measurable spaces and kernels to model the following situations: (Some parameters will be left unspecified)

(a) You roll a die to determine how often you will flip a coin; in the second stage, count the heads.

(b) You randomly select a street, then note the sex of the first person you meet.

(c) You randomly choose a passing car, then count the number of cars passing until a car of the same colour passes by.

(d) You choose a random point $X = (X_1, \dots, X_n)$ in $\mathbb{R}^d$ (after some probability measure μ, say) then you choose randomly and uniformly distributed a point from the shifted unit interval with X as center.

(e) (Cox-Ross-Rubinstein model of stock prices) A stock starts at a fixed value A; further, d, u are fixed numbers with $d < 1 < u$, and $p \in [0, 1]$ is fixed. Each day, a biased coin (probability of getting head is p) is thrown; if it is 'heads', the new price is u times the old price; if it is 'tails', the new price is d times the old price. (I.e., the first day you'll get either $u \cdot A$ or $d \cdot A$, depending on your coin toss.)

(Always start with identifying a 'reasonable' space.)

2. Consider a measure space $(\Omega, \mathfrak{A}, \mu)$ with a σ-finite measure μ and a function $f \in \overline{\mathfrak{Z}}_+(\Omega, \mathfrak{A})$. Show that

$$\int f \, d\mu = \int_{]0,\infty[} \mu(\{f > x\}) \, \lambda_1(dx).$$

(*Hint:* Start with the right hand side, use Fubini.)

3. Consider a queue where, per time step,

- in case of a non-empty queue, the customer at the head of the queue is served and leaves,
- n new customers arrive with probability b_n for $n \in \mathbb{N}_0$.

a) Choose an appropriate measurable space to model the lengths of the queue at all times $i \in \mathbb{N}_0$. Define the corresponding transition kernel by means of an infinite-dimensional stochastic matrix $(\overline{K}(k, \ell))_{k,\ell \in \mathbb{N}_0}$, i.e., $\overline{K}(k, \ell) \geq 0$ for $k, \ell \in \mathbb{N}_0$ and $\sum_{\ell=0}^{\infty} \overline{K}(k, \ell) = 1$ for $k \in \mathbb{N}_0$.

b) Suppose that initially the queue is empty. Derive a recursive formula for the probability of length $k \in \mathbb{N}_0$ of the queue at time $i \in \mathbb{N}_0$. Derive a formula for the probability of lengths $(k_1, \ldots, k_i)$ of the queue at times $1, \ldots, i$.

4. (*) The spherical random walk Let $\emptyset \neq D \subseteq \mathbb{R}^2$ be a bounded open set, for $x \in \mathbb{R}^2$, set $d(x, \partial D) := \inf\{|x - y|_2 \ : \ y \in \partial D\}$.($|\cdot|_2$ denotes Euclidean distance, and $\partial D = \bar{D} \cap \overline{D^c}$). Let μ denote the uniform distribution on the unit circle C (we will construct this in the next week, sorry, but you know what I mean). We describe a random experiment: Fix x in the interior of D, and draw the circle $x + C \cdot d(x, \partial D)$. Select randomly (i.e., uniformly) one point x_1 on this unit circle. Replace x with x_1, and iterate this procedure, indefinitely (alternately: Until $d(x_n, \partial D) < \varepsilon_0$), creating a random sequence of points $x_1, x_2, \ldots$

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(a) Make a stochastic model for the randomly built sequence $(x_1, x_2, \ldots)$ by finding appropriate kernels. Do you need additional assumptions for making the theory run?

(b) Write a computer program which simulates a spherical random walk for D being the unit sphere[1], for the unit cube, and for a triangle. Stop the process if $d(x_n, D) < 10^{-4}$, select some y 'close' to x_n on ∂D. Try to get, by experiment, a rough idea how the distribution of y looks like; do you, in particular, get a uniform distribution on the unit circle?

Remark: One can show that for suitable D, within the model constructed in (a), the series $(\xi_i)_{i \in \mathbb{N}}$ of random points converges almost surely to a random point η_x on ∂D, depending on the starting point x; moreover, if $f : \partial D \to \mathbb{R}$ is a nice function, then η has a connection to the Dirichlet problem given by f; namely, if u is the solution of

$$\Delta u = 0 \text{ on } D, \qquad u = f \text{ on } \partial D \,,$$

then $\mathbb{E}\, f(\eta_x) = u(x)$.

[1] don't start at 0