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5. Aufgabenblatt zur Vorlesung
'Probability Theory'

In the sequel, $(\Omega, \mathfrak{A})$ is a measure space.

1. (warming up) Let μ, ν, κ be σ–finite measures on $(\Omega, \mathfrak{A})$.

(i) Show that, if $\mu \ll \nu$ and $\nu \ll \kappa$, then $\mu \ll \kappa$. Further, if $\mu = f \cdot \nu$ and $\nu = g \cdot \kappa$, find the κ–density of μ.

(ii) Give an example of probability measures μ, ν such that $\mu \ll \nu \ll \mu$, but $\mu \neq \nu$.

(iii) Find the μ_1–density of μ_2:

(a) $\mu_2 = \lambda_1$ (Lebesgue measure),

$$\mu_1(A) = 1/2 \cdot \lambda_1(A \cap]-\infty, 0]) + 2 \cdot \lambda_1(A \cap [0, \infty[) \ .$$

(b)

$$\mu_i(A) = \int_{A \cap [0,\infty[} e^{-\alpha_i x} \, \mathrm{d}x \ ,$$

$(\alpha_1, \alpha_2 > 0$ fixed).

2. In this exercise, we study **Lebesgue's Decomposition Theorem:**

Let μ, ν be σ–finite measures on $(\Omega, \mathfrak{A})$. Then there exist σ–finite measures μ_a, μ_s such that $\mu = \mu_a + \mu_s$, $\mu_a \ll \nu$ and $\mu_s(N^c) = 0$ for some $N \in \mathfrak{A}$ with $\nu(N) = 0$.

This means that we can dissect μ in a part which is absolutely continuous w.r.t. ν, and a part which 'lives' only on a set of ν–measure zero.

(i) Since $\mu, \nu \ll \mu + \nu$, we can find densities p, q:

$$\mu = p \cdot (\mu + \nu), \qquad \nu = q \cdot (\mu + \nu) \ .$$

Denote

$$f := \mathbb{1}_{\{q>0\}} \cdot p/q, \qquad N := \{q = 0\} \ .$$

Show that

$$\mu(A) = \mu(A \cap N) + \int_A f \, \mathrm{d}\nu \qquad \forall A \in \mathfrak{A} \ . \tag{1}$$

Why does this imply Lebesgue's Decomposition Theorem?

(ii)(*) Show that the decomposition is essentially unique, i.e., if you can find (f_1, N_1) and (f_2, N_2) with the property (1), then $f_1 = f_2$ ν–a.e., and $\mu(N_1 \Delta N_2) = 0$.
(*Hint:* Prove first that $\int_A g_1 \mathrm{d}\nu = \int_A g_2 \mathrm{d}\nu$ for $g_i := f_i \cdot \mathbb{1}_{N_1^c \cap N_2^c}$, and apply Theorem 2.8.3.)

(iii) Find the Lebesgue decomposition of μ with respect to ν:

(a) $\mu = Pois(\lambda)$ (Poisson distribution), $\nu = B(n,p)$ (binomial distribution),

(b) $\mu = Cau(1)$ (Cauchy distribution, μ hat Lebesgue density $f(x) = \frac{1}{\pi(1+x^2)}$), $\nu = Exp(\lambda)$ (exponential distribution, Lebesgue density $g(x) = \mathbb{1}_{x \geq 0} \lambda e^{-\lambda x}$).

(*) Can you formulate a general rule for obtaining the Lebesgue decomposition in such cases?

3. On the space $\mathcal{M} = \mathcal{M}(\Omega, \mathfrak{A})$ of all measures over $(\Omega, \mathfrak{A})$, consider the relation

$$\mu \sim \nu \quad :\Leftrightarrow \quad \mu \ll \nu \ll \mu \, .$$

(i) Show that this is an equivalence relation.

(ii) Is it possible to define addition and positive scalar multiplication on the factor space $\mathcal{M}/\sim$ of equivalence classes by setting

$$a[\mu]_\sim \; + \; b[\nu]_\sim := [a\mu + b\nu]_\sim \, , \qquad \mu, \nu \in \mathcal{M}, a, b \geq 0 \, ?$$

(iii) Prove or disprove: $\mu \sim \nu$ if and only if there is a measurable $p : \Omega \to]0, \infty]$ (strictly positive!) such that $\mu = p \cdot \nu$.