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4. Aufgabenblatt zur Vorlesung "'Probability Theory"'

1. (Warming up) a) Consider a probability measure P on $(\mathbb{R}, \mathcal{B})$ and define the cumulative distribution function (cdf) $F : \mathbb{R} \to [0,1]$ by $F(x) = P(]-\infty, x])$. Show that

(i) F is non-decreasing,

(ii) F is right continuous,

(iii) $\lim_{x\to\infty} F(x) = 1$, $\lim_{x\to-\infty} F(x) = 0$,

(iv) for all $x \in \mathbb{R}$, $\lim_{y\downarrow x} F(y) - \lim_{y\uparrow x} F(y) = P(\{x\})$. In particular, F is continuous at x iff $P(\{x\}) = 0$.

b) Consider a function $F : \mathbb{R} \to [0,1]$ that satisfies (i)–(iii). Sketch the proof of the following fact: There exists a uniquely determined probability measure P on $(\mathbb{R}, \mathcal{B})$ such that $F(x) = P(]-\infty, x])$ for every $x \in \mathbb{R}$.

2. Consider the stochastic model $(\Omega, \mathcal{A}, P)$ for coin tossing with an infinite sequence of trials, see Remark II.4.3.(ii).

a) Show that $\{\omega\} \in \mathcal{A}$ and $P(\{\omega\}) = 0$ for every $\omega \in \Omega$.

b) Let $S_1, \ldots, S_n$ denote pairwise disjoint sets in $\mathcal{P}_0(\mathbb{N})$ and let $A_j \in \mathcal{P}(\{0,1\}^{S_j})$. Show that

$$P\Big(\bigcap_{j=1}^{n} \pi_{S_j}^{-1}(A_j)\Big) = \prod_{j=1}^{n} P(\pi_{S_j}^{-1}(A_j)).$$

c)(*) Let $S_j = \{1\}$ and $S_j = \{\binom{j}{2} + 1, \ldots, \binom{j+1}{2}\}$ for $j \geq 2$. Show that

$$0 < P\big(\{\omega \in \Omega : \forall\, j \in \mathbb{N}\ \exists\, i \in S_j :\ w_i = 0\}\big) < 1.$$

(*Hint:* Use b) and σ–continuity to write the probability as an infinite product, use $x = \exp(\log x), x > 0$ to express this by an infinite sum, and use $c_1 x \leq \log(1+x) \leq c_2 x$ for $x \in [-1,1]$.)

3. Consider the measure space $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu)$, where μ is the counting measure.

a) Let $f : \mathbb{N} \to \mathbb{R}$. Show that

$$\sum_{n\geq 1} f(n) \text{ converges absolutely} \quad \text{iff} \quad f \text{ is } \mu\text{-integrable.}$$

In this case

$$\int f\, d\mu = \sum_{n\geq 1} f(n).$$

b) Formulate the 'dominated convergence theorem' in this particular situation.

4. Let $(\Omega, \mathcal{A}, \mu)$ be a measure space. Show that if $f \in \mathcal{L}^1(\Omega, \mathcal{A}, \mu)$, then for each $\varepsilon > 0$ there is a $\delta > 0$ such that $\mu(A) < \delta$ implies $\int_A |f| \, d\mu < \varepsilon$.
(*Hint:* Indirect proof, use Lebesgue's theorem.)