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3. Aufgabenblatt zur Vorlesung
"'Probability Theory"'

1. Let $T > 0$ and consider the product space $(\mathbb{R}^{[0,T]}, \bigotimes_{t\in[0,T]} \mathcal{B})$. Let C denote the set of real-valued continuous functions on $[0,T]$. Show that $A \in \bigotimes_{t\in[0,T]} \mathcal{B}$ and $A \subset C$ implies that $A = \emptyset$. Thus, in particular $C \notin \bigotimes_{t\in[0,T]} \mathcal{B}$. Show, on the other hand, that $A \in \bigotimes_{t\in[0,T]} \mathcal{B}$ and $C \subset A$ do not imply that $A = \mathbb{R}^{[0,T]}$.

2. a) Find a content μ on an algebra $\mathfrak{A}$ which is σ–continuous from above, but not from below.
Hint: It is most convenient to use a content which attains only the values $0, \infty$.

b) Find a measure space $(\Omega, \mathfrak{A}, \mu)$ and $A_1, A_2, \ldots \in \mathfrak{A}$ such that $A_n \downarrow \emptyset$, but $\lim_{n\to\infty} \mu(A_n) \neq 0$.

3. Let the probability space $(\Omega, \mathcal{A}, P)$ be *nonatomic*, i.e., $P(A) > 0, A \in \mathcal{A}$, implies that there exists a $B \in \mathcal{A}$ such that $B \subset A$ and $0 < P(B) < P(A)$.

a) Show that $P(A) > 0, A \in \mathcal{A}$, and $\varepsilon > 0$ imply that there exists a $B \in \mathcal{A}$ such that $B \subset A$ and $0 < P(B) < \varepsilon$.
Hint: Indirect proof, assume the assumption to be wrong. Define a descending sequence $B_n \subseteq B_{n-1} \subseteq A$ with $P(B_n)$ ‚almost minimal' in B_{n-1}. Show that $B = \bigcap_n B_n$ is an atom.

b) Show that there exists an $A \in \mathcal{A}$ such that $P(A) = \frac{1}{2}$ (In fact, for each $x \in [0,1]$ there exists an $A \in \mathcal{A}$ such that $P(A) = x$).
Hint: Show first, in a slight variation of part a), that each subset A of measure $P(A) > 1/2$ contains measurable subsets B_ε with $1/2 \leq P(B) < 1/2 + \varepsilon$ for any $\varepsilon \in (0, P(A) - 1/2)$. Using this, find then a descending sequence B_n of sets with $1/2 \leq P(B_n) \leq 1/2 + 1/n$.

c) Show for nonatomic measures P_1 and P_2 on $(\Omega, \mathcal{A})$ that there exists an $A \in \mathcal{A}$ such that $P_1(A) \geq \frac{1}{2}$ and $P_2(A^c) \geq \frac{1}{2}$
Hint: This is exactly the classical problem of dividing a piece of cake 'fairly' between two children.

Remark: The nonatomicity of the measures asserts e.g. that there is no single candy on top of the cake. Then the problem is impossible to solve, as will any parent tell you, hence the assumption is necessary.