Seminar
Numerik und
Wissenschaftliches Rechnen

Chapter 2:
A Simple Example

Michael Kreim
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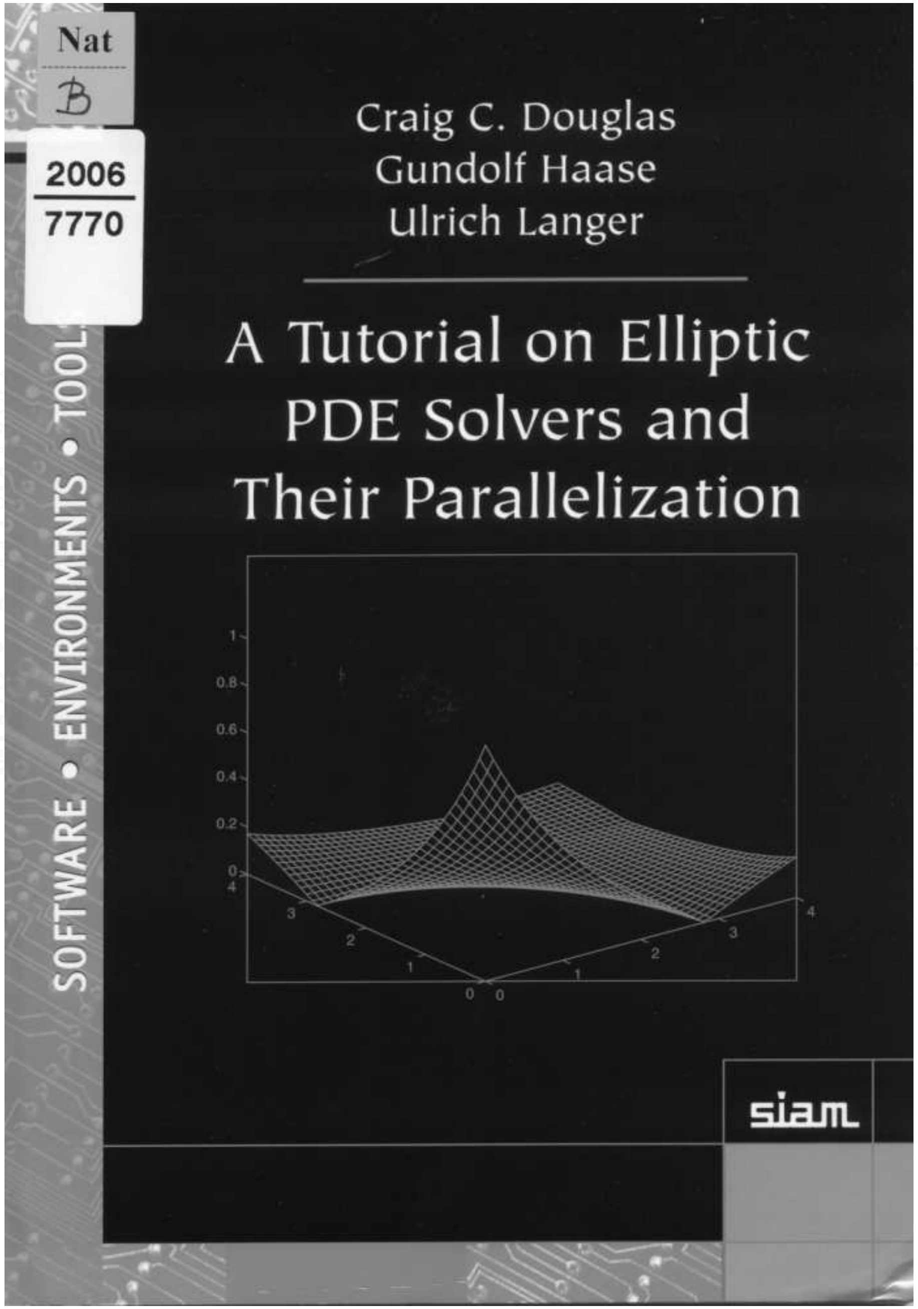


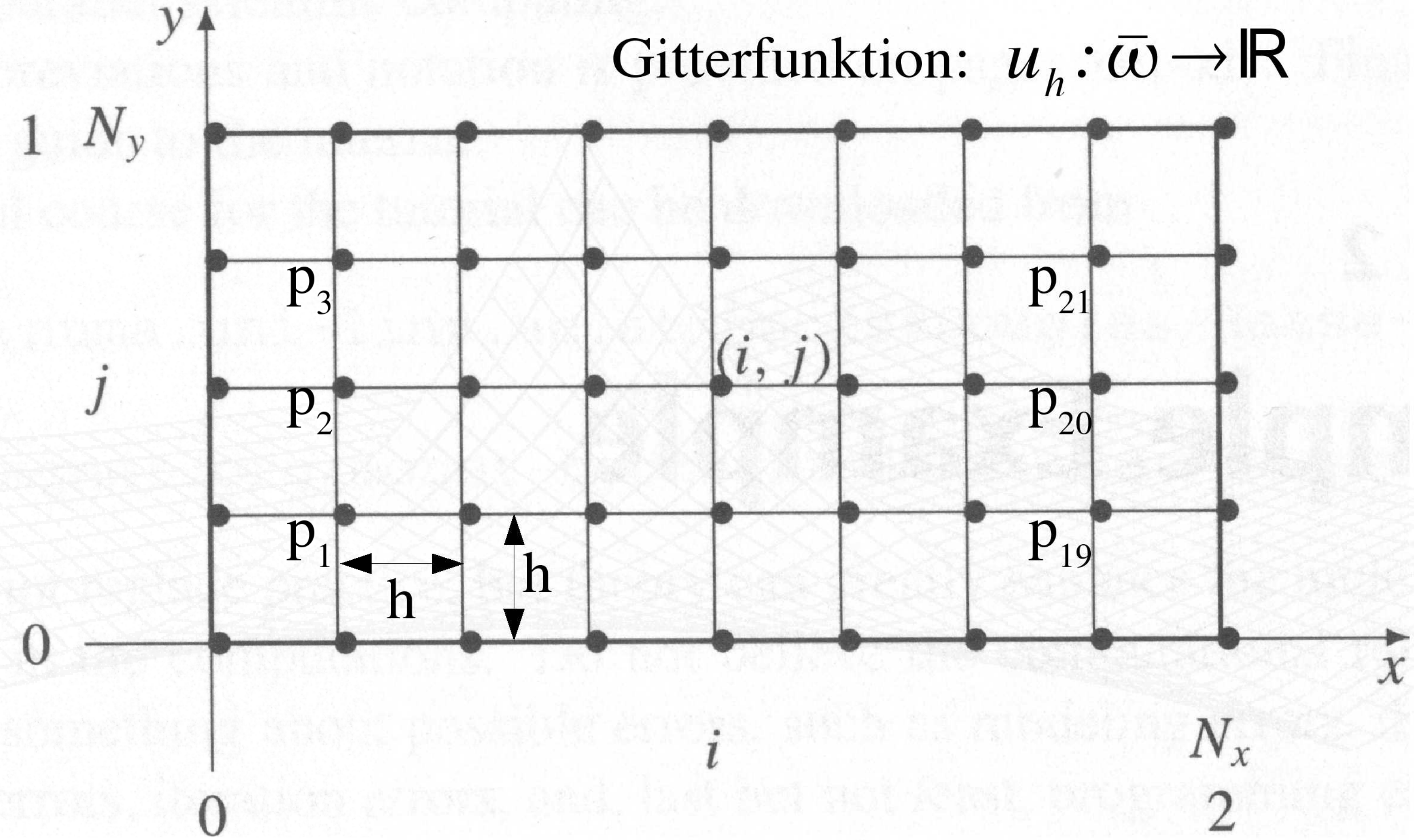
Poisson Gleichung

$$-\Delta u(x,y)=f(x,y) \qquad \forall (x,y)\in\Omega$$

$$u(x,y)=g(x,y) \qquad \forall (x,y)\in\partial\Omega$$

- = (0,2) x (0,1)
- Dirichlet Rand-Bedingungen: homogen
- finde $u:\bar{\Omega}:=\Omega\cup\partial\Omega\to\mathbb{R}$

Definition eines Gitters



Gitterfunktion: $u_h : \bar{\omega} \to \mathbb{R}$

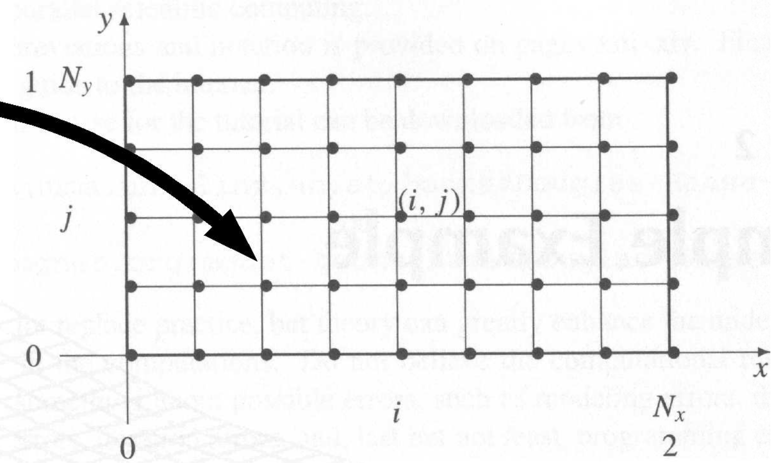
Knoten $\bar{\omega}_h := \{(i, j): \; i = 0, N_x \;\; j = 0, N_y\}$

Innere Knoten $\omega_h := \{(i, j): \; i = 0, N_x - 1 \;\; j = 0, N_y - 1\}$

Randknoten $\gamma_h := \bar{\omega}_h \setminus \omega_h$

Finite Differenzen Methode

$$\frac{1}{h^2}\begin{bmatrix} & -1 & \\ -1 & 4 & -1 \\ & -1 & \end{bmatrix}$$



$$\frac{1}{h^2}\left(-u_{i,j-1}-u_{i-1,j}+4u_{i,j}-u_{i+1,j}-u_{i,j+1}\right)=f_{i,j} \qquad \forall (i,j)\in\omega_h$$

$$u_{i,j}=0 \qquad \forall (i,j)\in\gamma_h$$

$$K_h U_h = f_h$$

$$\left[\begin{array}{ccc|ccc|ccc} -1 & & & 4 & -1 & & -1 & & \\ & -1 & & -1 & 4 & -1 & & -1 & \\ & & -1 & & -1 & 4 & & & -1 \end{array}\right]$$

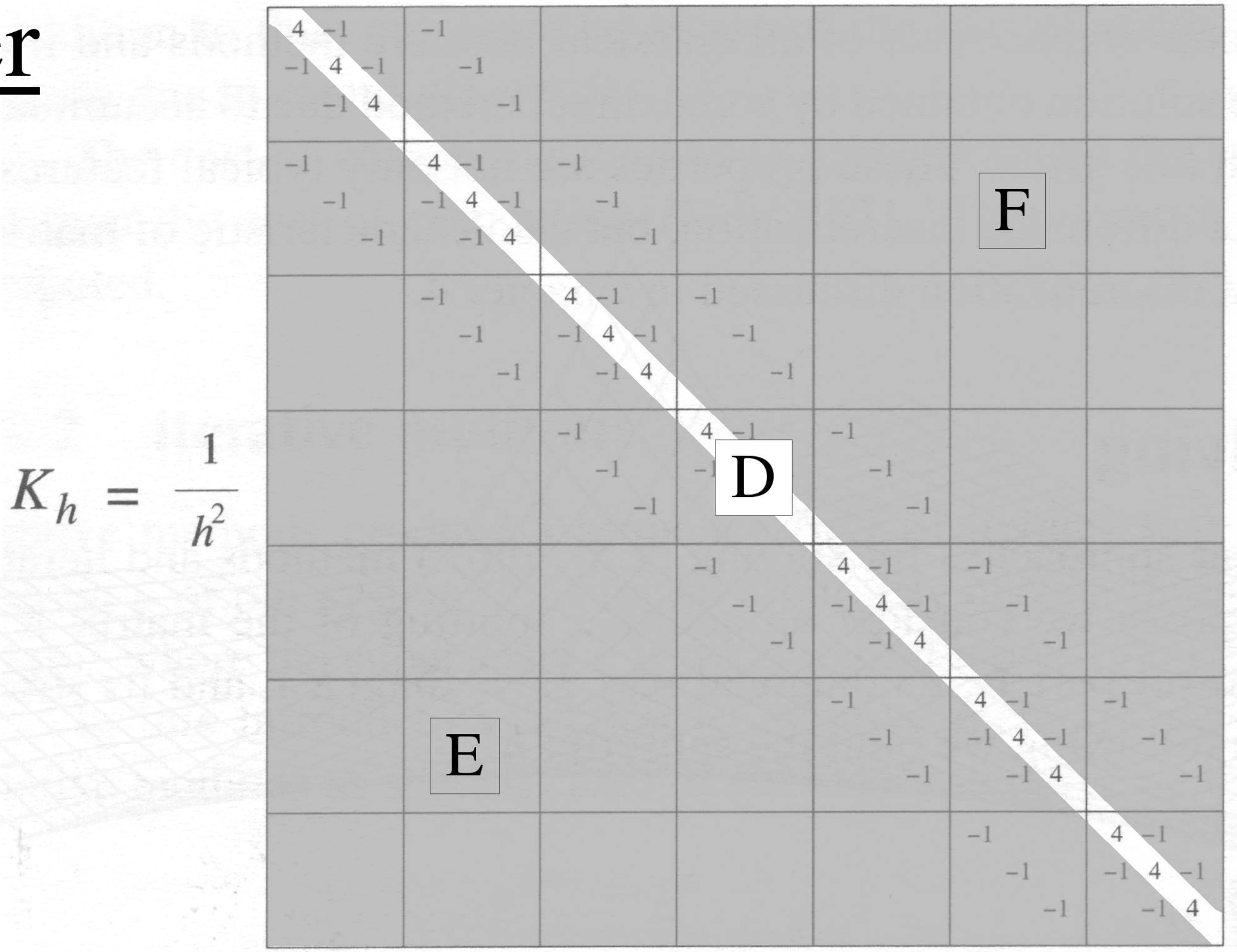
$$h = \frac{1}{h^2}\left[\begin{array}{c|c|c|c|c|c|c}
\begin{smallmatrix}4 & -1 & \end{smallmatrix} & \begin{smallmatrix}-1 & & \end{smallmatrix} & & & & & \\ \hline
 & & \begin{smallmatrix}-1 & & \\ & -1 & \\ & & -1\end{smallmatrix} & \begin{smallmatrix}4 & -1 & \\ -1 & 4 & -1 \\ & -1 & 4\end{smallmatrix} & \begin{smallmatrix}-1 & & \\ & -1 & \\ & & -1\end{smallmatrix} & & \\ \hline
 & & & \begin{smallmatrix}-1 & & \\ & -1 & \\ & & -1\end{smallmatrix} & \begin{smallmatrix}4 & -1 & \\ -1 & 4 & -1 \\ & -1 & 4\end{smallmatrix} & \begin{smallmatrix}-1 & & \\ & -1 & \\ & & -1\end{smallmatrix} & \\ \hline
 & & & & \begin{smallmatrix}-1 & & \\ & -1 & \\ & & -1\end{smallmatrix} & \begin{smallmatrix}4 & -1 & \\ -1 & 4 & -1 \\ & -1 & 4\end{smallmatrix} & \begin{smallmatrix}-1 & & \\ & -1 & \\ & & -1\end{smallmatrix} \\ \hline
 & & & & & \begin{smallmatrix}-1 & & \\ & -1 & \\ & & -1\end{smallmatrix} & \begin{smallmatrix}4 & -1 & \\ -1 & 4 & -1 \\ & -1 & 4\end{smallmatrix}
\end{array}\right]
\qquad
\underline{f}_h = \begin{bmatrix} f_{1,1} \\ f_{1,2} \\ f_{1,3} \\ f_{2,1} \\ f_{2,2} \\ f_{2,3} \\ f_{3,1} \\ f_{3,2} \\ f_{3,3} \\ f_{4,1} \\ f_{4,2} \\ f_{4,3} \\ f_{5,1} \\ f_{5,2} \\ f_{5,3} \\ f_{6,1} \\ f_{6,2} \\ f_{6,3} \\ f_{7,1} \\ f_{7,2} \\ f_{7,3} \end{bmatrix}$$

Eigenschaften des Systems

- K_h ist sparse (hier max. 5 Einträge / Zeile)
- K_h ist spd ($K_h = K_h^T$; $\langle K_h U_h, U_h \rangle > 0$)
- Dimension N des Systems wächst mit $O(h^{-m})$ m=Dim( )
- K_h ist schlecht konditioniert:

$$\kappa(K_h) = \frac{\lambda_{max}(K_h)}{\lambda_{min}(K_h)} = O(h^{-2})$$

Löser



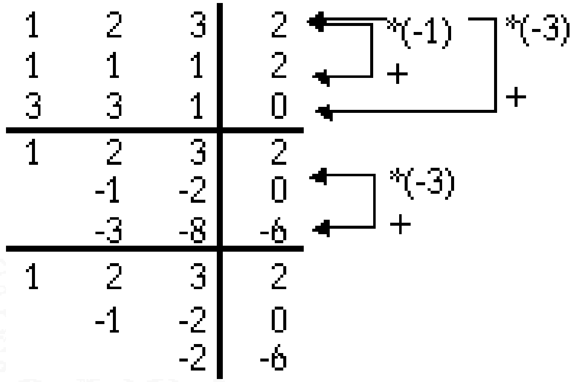
$$K = E + D + F$$

direkte Lösungsmethoden

- Gaußsches Eliminationsverfahren

  → $E_{i,j}=0$

$$\left(\tilde{D}+\tilde{F}\right)U=\tilde{f}$$



- LR-Zerlegung

$$K=LR$$

$$Lx=f$$

$$RU=x$$

K ist spd

=> Cholesky-Verfahren

Probleme bei direkten Lösungsmethoden

- Zunahme der Komplexität
  - arithmetische Operationen $O(h^{-3m+2})$
  - Speicherbedarf $O(h^{-2m+1})$
- Verlust von $\log(\kappa(K))$ Nachkommastellen durch Rundungsfehler

iterative Lösungsmethoden

$$U^0 \in \mathbb{R}^n \to \{ U^k \} \underset{k \to \infty}{\to} U$$

$$C \frac{U^{k+1} - U^k}{\tau} + K U^k = f$$

Relaxationsparameter (τ)

Vorkonditionierer (C)

hier: C,K spd

-> Konvergenz für $\tau \in \left(0 \; , \; \frac{2}{\lambda_{max}(C^{-1} K)} \right)$

Beispiele für iterative Lösungsmethoden

- Richardson Iteration (C:=I)

$$U^{k+1}=U^k+\tau(f-KU^k)$$

- ω-Jacobi-Iteration (τ:=ω , C:=D)

$$U^{k+1}=U^k+\omega D^{-1}(f-KU^k)$$

- vorwärts/rückwärts Gauss-Seidel-Iteration (C=D+E)/(C=D+F)

$$(D+E)U^{k+1}=f-FU^k$$

$$(D+F)U^{k+1}=f-EU^k$$

Beispiele für iterative Lösungsmethoden

- forward/backward successive overrelaxation (SOR)

$$\tau=\omega; \quad C=D+\omega E \quad oder \quad C=D+\omega F$$

$$(D+\omega E)U^{k+1}=(1-\omega)DU^k+\omega(f-FU^k)$$

$$(D+\omega F)U^{k+1}=(1-\omega)DU^k+\omega(f-EU^k)$$

- symmetric succesive overrelaxation iteration (SSOR)

  Abwechselnd

  -> forward SOR

  -> backward SOR

$$\tau=\omega(2-\omega)$$

$$\omega\in(0,2)$$

$$C=(D+\omega E)D^{-1}(D+\omega F)$$

Beispiele für iterative Lösungsmethoden

- alternating direction implicit iterative method (ADI)

$$-\Delta u \quad = \quad -u_{xx} \quad -u_{yy}$$

$$\begin{bmatrix} & -1 & \\ -1 & 4 & -1 \\ & -1 & \end{bmatrix} = \begin{bmatrix} & & \\ -1 & 2 & -1 \\ & & \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}$$

$$K \quad = \quad K\,x \quad + \quad K\,y$$

$$(K_x + K_y)U = (K_x + \rho I)U + (K_y - \rho I)U = f$$

$$(K_x + \rho I)U = f - (K_y - \rho I)U$$

$$(K_x + \rho I)U^{l+1} = f - (K_x + K_y)U^l + (K_x + \rho I)U^l$$

=> Iteration in x-Richtung

$$(K_x + \rho I)(U^{l+1} - U^l) = f - KU^l$$

Beispiele für iterative Lösungsmethoden

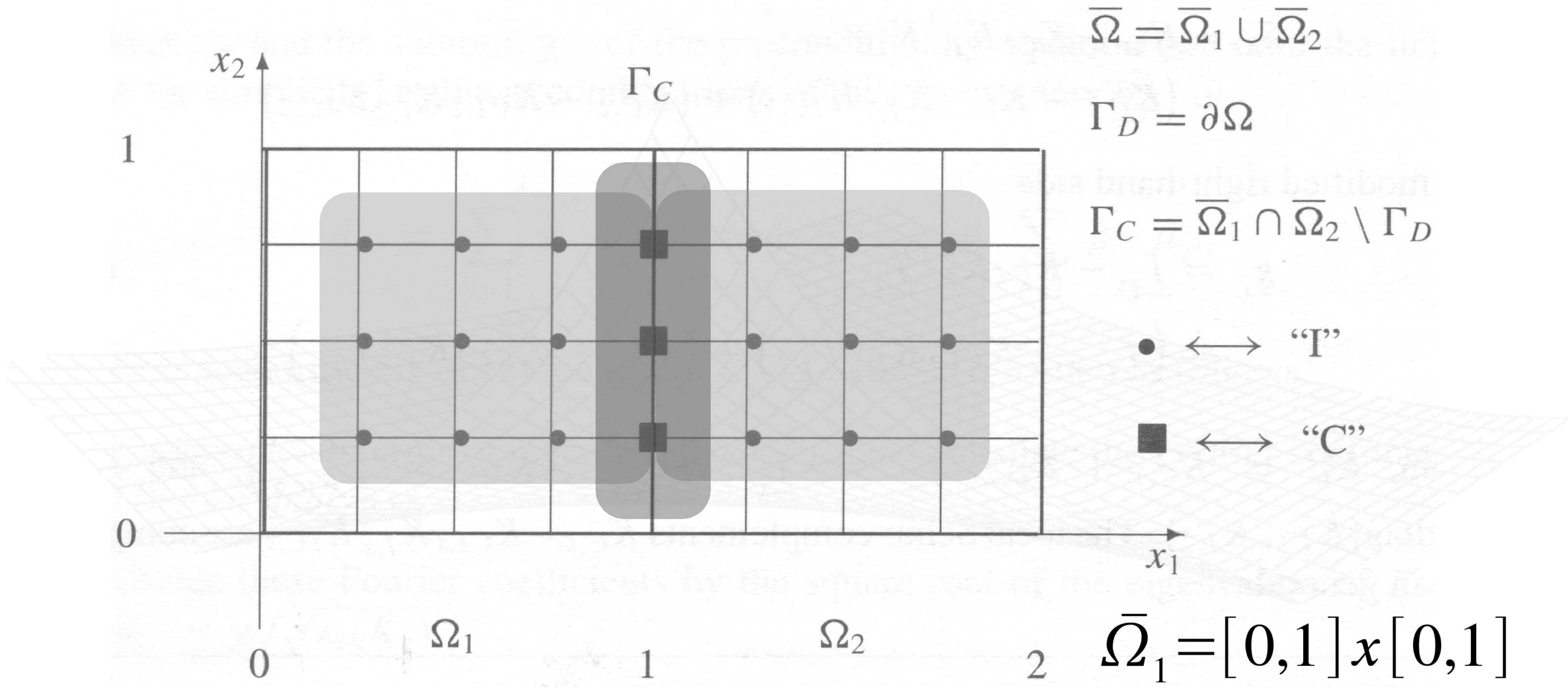
- alternating direction implicit iterative method (ADI)

$$(K_x + \rho I) U^{k+\frac{1}{2}} = f - (K_y - \rho I) U^k$$

$$(K_y + \rho I) U^{k+1} = f - (K_x - \rho I) U^{k+\frac{1}{2}}$$

-> Vortrag am 06. 02.07 Sebastian Plitzko ADI

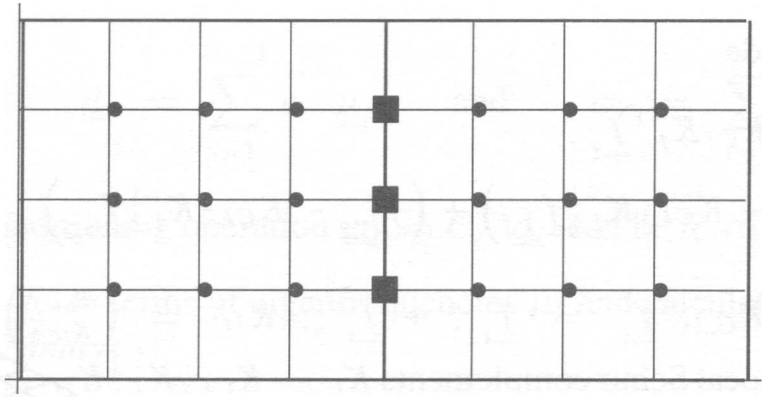
Domain Decomposition (DD)



$$\bar{\Omega}_1 = [0,1] x [0,1]$$
$$\bar{\Omega}_2 = [1,2] x [0,1]$$

Domain Decomposition (DD)

$$\begin{pmatrix} K_C & K_{CI,1} & K_{CI,2} \\ K_{IC,1} & K_{I,1} & 0 \\ K_{IC,2} & 0 & K_{I,2} \end{pmatrix} \begin{pmatrix} U_C \\ U_{I,1} \\ U_{I,2} \end{pmatrix} = \begin{pmatrix} f_C \\ f_{I,1} \\ f_{I,2} \end{pmatrix}$$



$$\begin{pmatrix} S_C & 0 \\ K_{IC} & K_I \end{pmatrix} \begin{pmatrix} U_C \\ U_I \end{pmatrix} = \begin{pmatrix} g_C \\ f_I \end{pmatrix}$$

$$S_C = K_C - K_{CI} K_I^{-1} K_{IC}$$

$$g_C = f_C - K_{CI} K_I^{-1} f_I$$

Domain Decomposition (DD)

I) $\underline{g}_C := \underline{f}_C$

for $s := 1$ to P do in parallel

$$\underline{g}_C := \underline{g}_C - K_{CI,s} \cdot K_{I,s}^{-1} \cdot \underline{f}_{I,s}$$

end

II) Solve $S_C \, \underline{u}_C = \underline{g}_C$

III) for $s := 1$ to P do in parallel

$$\underline{u}_{I,s} := K_{I,s}^{-1} \cdot \left(\underline{f}_{I,s} - K_{IC,s} \cdot \underline{u}_C \right)$$

end