V euklidischer Vektorraum $\dim V = n$

Basis α

$$\det_\alpha(\vec a_1,\dots,\vec a_n) := \det(A)$$

$$A = \left(\vec a_1^{\,\alpha}, \dots, \vec a_n^{\,\alpha}\right)$$

Satz. α, β Orthonormalbasen

$$|\det_\alpha(\vec a_1,\dots,\vec a_n)| = |\det_\beta(\vec a_1,\dots,\vec a_n)|$$

Beweis $B = \left(\vec a_1^{\,\beta}, \dots, \vec a_n^{\,\beta}\right)$

$$\Rightarrow \quad A = {}^\alpha T_\beta \, B$$

${}^\alpha T_\beta$ orthogonal, $|\det {}^\alpha T_\beta| = 1$

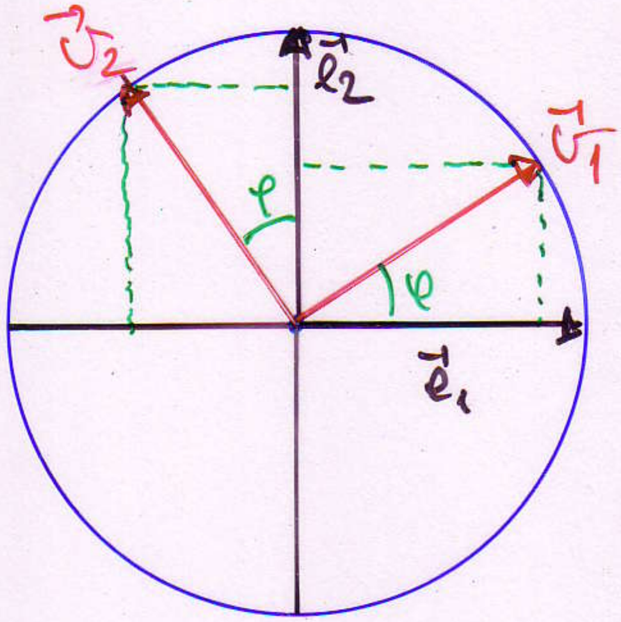
$$|\det A| = |\det {}^\alpha T_\beta \cdot \det B| = |\det B|$$

Korollar. $\vec a_1,\dots,\vec a_n$ orthogonal

$$\Rightarrow \det(\vec a_1,\dots,\vec a_n) = \|\vec a_1\| \cdot \ldots \cdot \|\vec a_n\|$$

Bew $\vec a_i = \|\vec a_i\| \vec e_i$, $\vec e_1,\dots,\vec e_n$ ON-Basis α

$$\det_\alpha(\vec e_1,\dots,\vec e_n) = 1$$



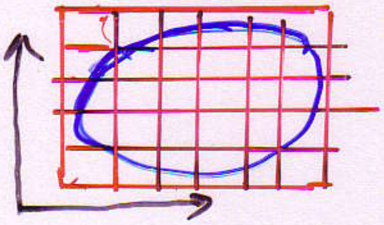
$\alpha : \vec{e}_1, \vec{e}_2 \qquad \beta : \vec{v}_1, \vec{v}_2$

$$^{\alpha}T_{\beta} = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$$

$$\det {}^{\alpha}T_{\beta} = \cos^2\varphi + \sin^2\varphi = 1$$

23.2. Unabhängigkeit des Jordan Maßes

Zu jedem kartesischen Koordinatensystem α ist Jordan-Maß μ_α über achsenparallele Rechtecke definiert



τ Verschiebung $\quad \alpha$

$\Rightarrow \mu_\alpha(\tau(B)) = \mu_\alpha(B)$

Satz 23.1 / Lemma 23.5

Für alle kart. Koo.syst. α, β gilt

B μ_α-messbar $\Leftrightarrow$ B μ_β-messbar

$\mu_\alpha(B) = \mu_\beta(B)$

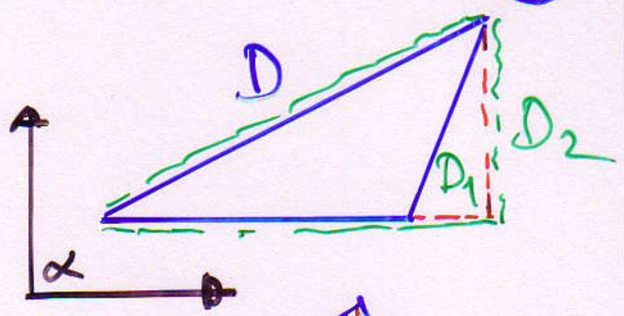
$B = \{ \sum_{i=1}^{n} r_i \vec{a}_i + O \mid r_i \in [0,1] \}$ Spat

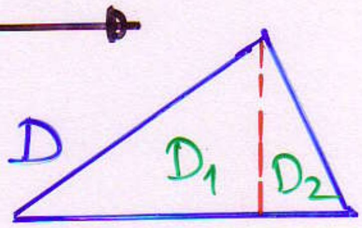
$\Rightarrow \mu(B) = |\det(\vec{a}_1, \ldots, \vec{a}_n)|$

V n-dimensionaler euklidischer Vektorraum $\cong \mathbb{R}^n$

$n = 2$

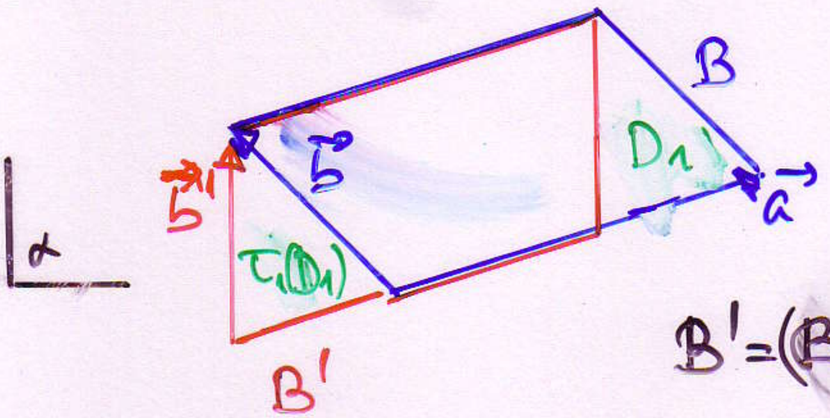
Hilfssatz D Dreieck mit α-achsenparalleler Seite $\Rightarrow$ D μ_α-messbar



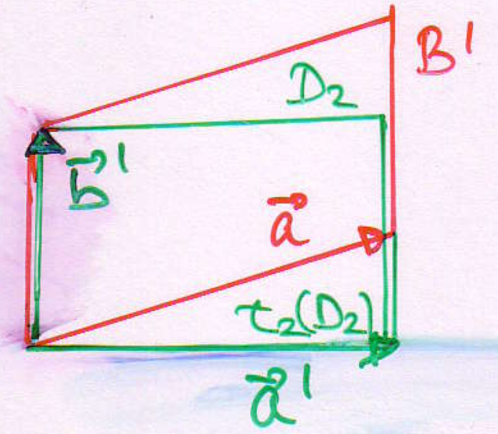


Bew: $D = D_2 \setminus D_1$

bzw $D = D_1 \cup D_2$ $\quad$ D_i messbar Übung



$B' = (B \setminus D_1) \cup \tau_1(D_1)$

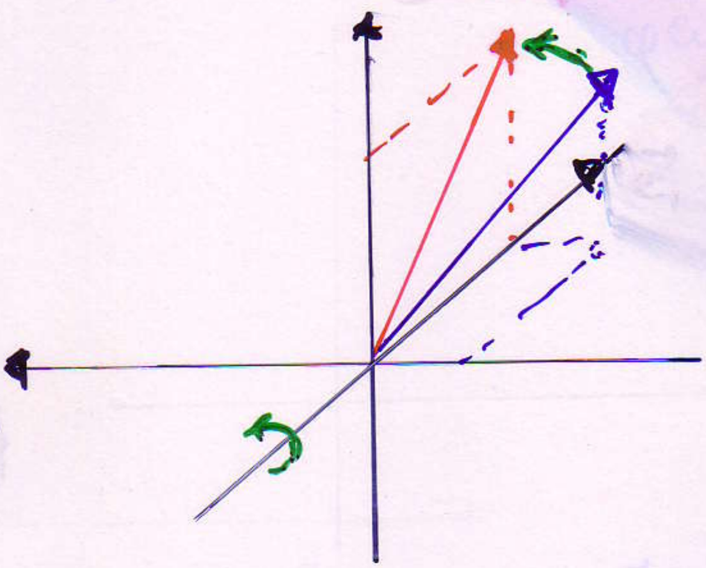


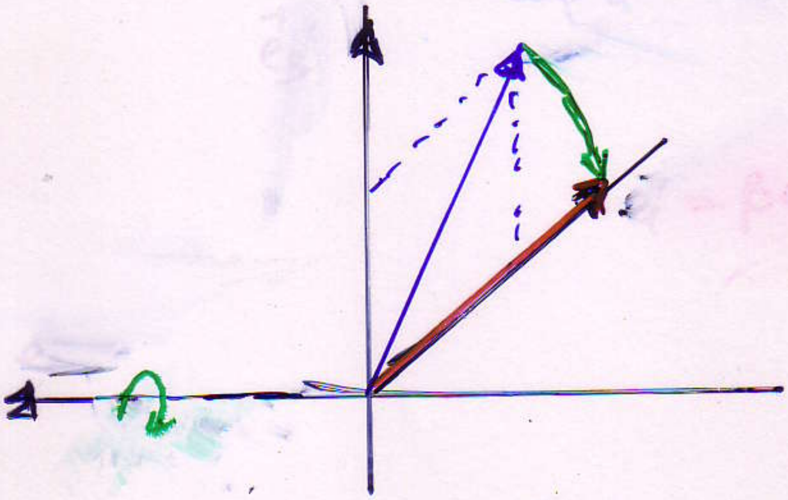
$B'' = (B' \setminus D_2) \cup \tau_2(D_2)$

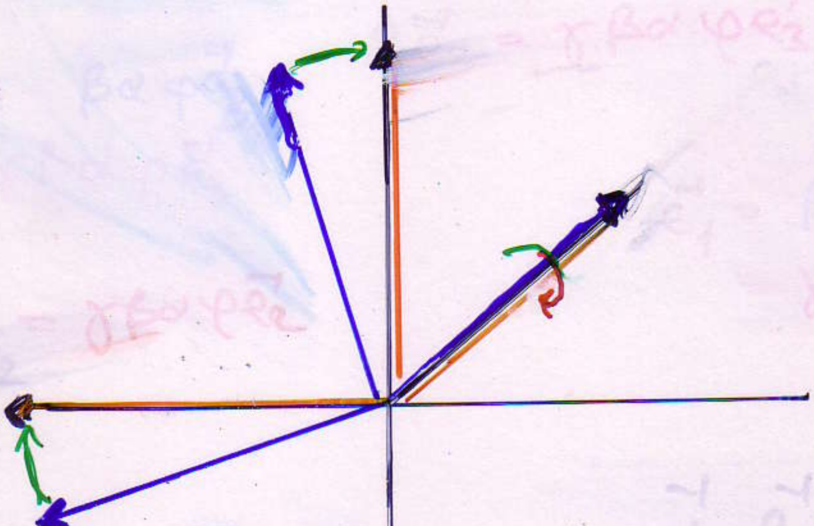
$$\mu(B) = \mu(B') = \mu(B'')$$

$$\det(\vec{a},\vec{b}) = \det(\vec{a},\vec{b}\,') = |\det(\vec{a}\,',\vec{b}\,')|$$

n=3 Euler







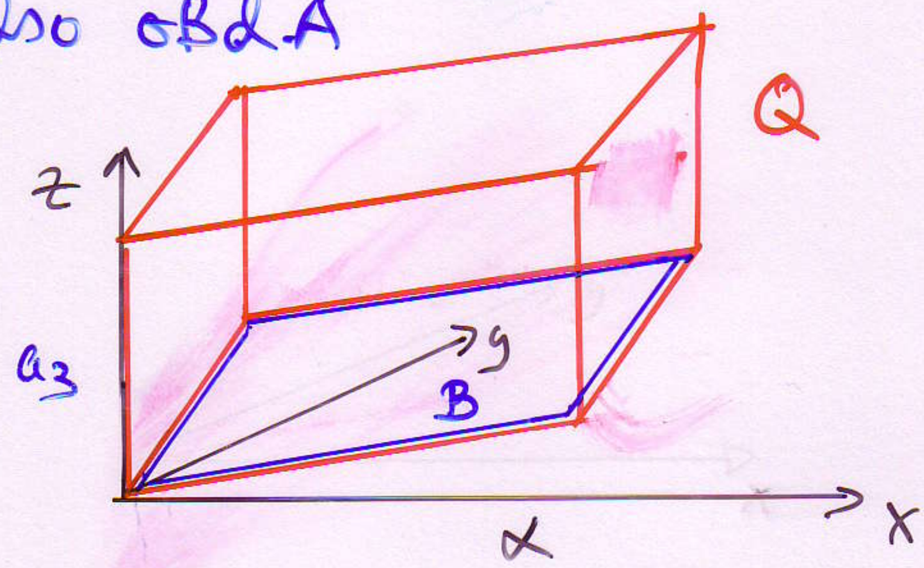
$\varphi = \alpha^{-1}\beta^{-1}\gamma^{-1}$

Euler α, β ONB

$\Rightarrow \exists\ \alpha = \beta_0, \dots, \beta_4 = \beta$ ONB

β_i, β_{i+1} haben gemeinsame Achse

also oBdA



$$\mu_\alpha(Q) = \int_B a_3 \, d(x,y) = a_3 \, \mu(B)$$
$$= a_3 a_1 a_2$$
$$= \mu_\beta(Q)$$

β achsenparallel Q

$\Rightarrow \mu_\alpha = \mu_\beta$

$S = \mathrm{Spat}(\vec{a}_1, \vec{a}_2, \vec{a}_3)$

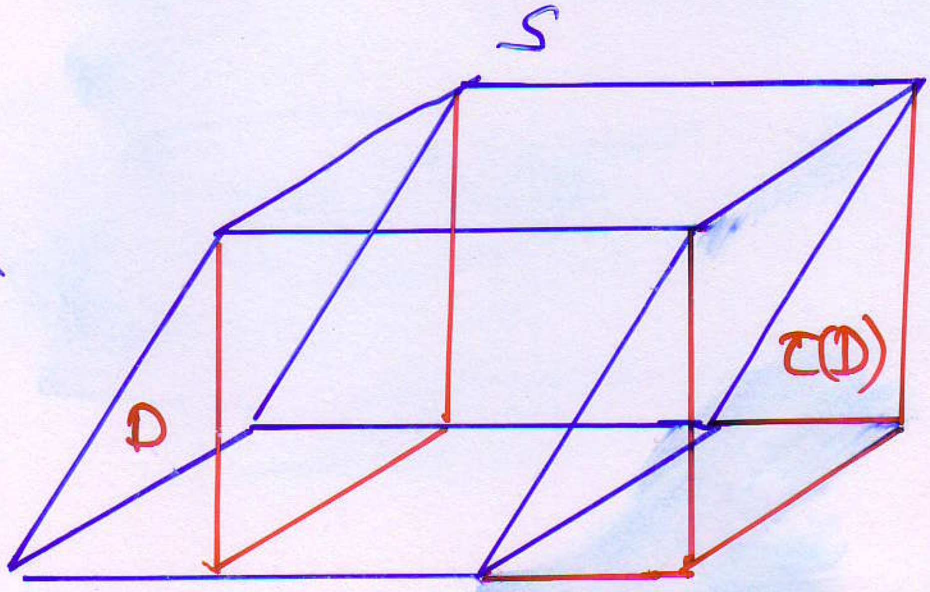
$\mu(S) = |\det A|$

Bew: $S \rightsquigarrow S'$ Vertauschung $\to S = S'$

$S \rightsquigarrow S'$ Scherung

$A \rightsquigarrow A'$ $\det A = \det A'$

$\mu(S) = \mu(S')$



τ Translation

D Normalbereich

$S' = (S \setminus D) \cup \tau(D)$

$\mu(S') = (\mu(S) - \mu(D)) + \mu(D)$

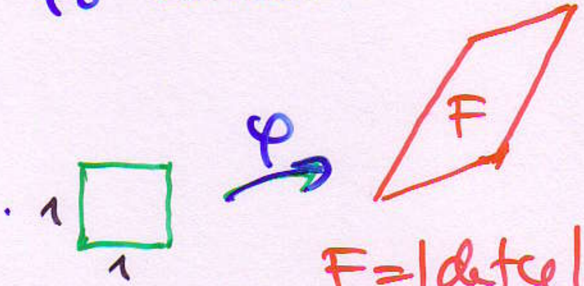
Nun $A \rightsquigarrow A''$ Diagonal $\det A'' = \det A$

$S \rightsquigarrow S''$ Quader $\overset{H}{\mu}(S'') = \mu(S)$

23.3.1 Affine Abb $\varphi : V \to W$

$\varphi(\vec{x}) = \varphi_0(\vec{x}) + \vec{c}$ φ_0 linear

$\varphi(\vec{x})^\alpha = A\vec{x}^\alpha + \vec{c}^\alpha$



$V = W$

$\det \varphi := \det A$ $F = |\det \varphi|$

unabh. von α nach Produktsatz

φ umkehrbar $\Leftrightarrow$ $\det \varphi \neq 0$

$\Rightarrow \varphi^{-1}$ affin

$\det {}^\beta T_\alpha A^\alpha T_\beta = \det A$

23.3.2 Bewegung

$\| \varphi(\vec{x}) - \varphi(\vec{y}) \| = \| \vec{x} - \vec{y} \|$

$\Leftrightarrow$ A orthogonal α ON Basis

$\varphi(\vec{x}) = \vec{x} + \vec{c}$ Verschiebung Translation

$\mathbb{R}^2$ Drehung, Spiegelung, Gleitspiegelung

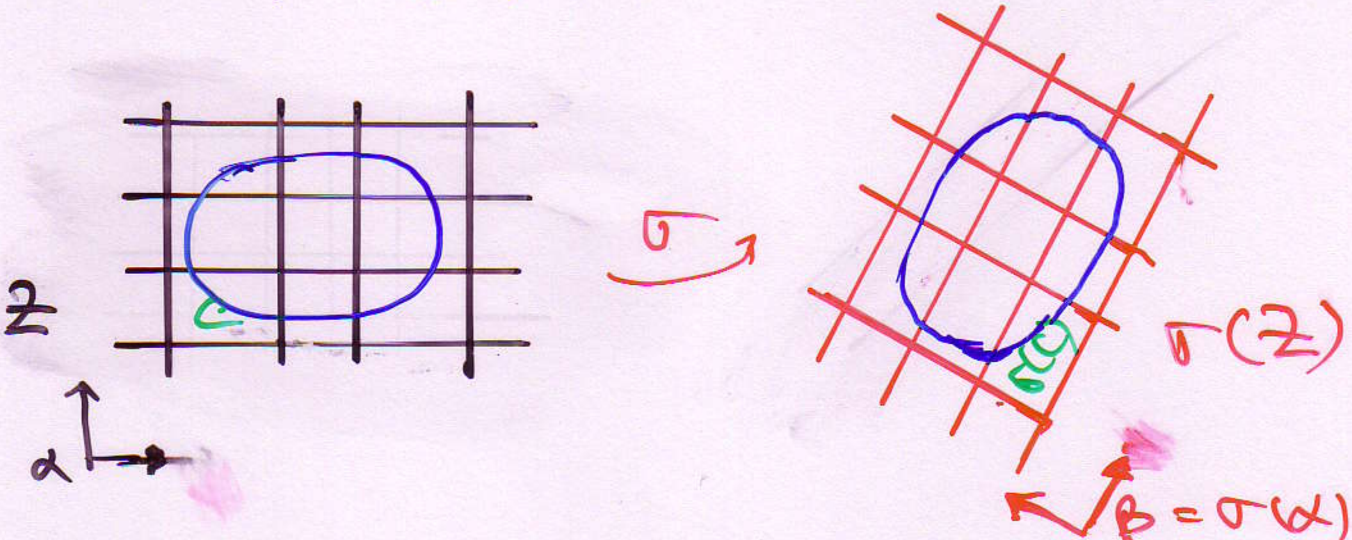
$\mathbb{R}^3$ Schraubung

23.4.1

σ Bewegung, B messbar

$\Rightarrow \sigma(B)$ messbar, $\mu(\sigma(B)) = \mu(B)$

$\sigma(Z) = \{\sigma(C) \mid C \in Z\}$



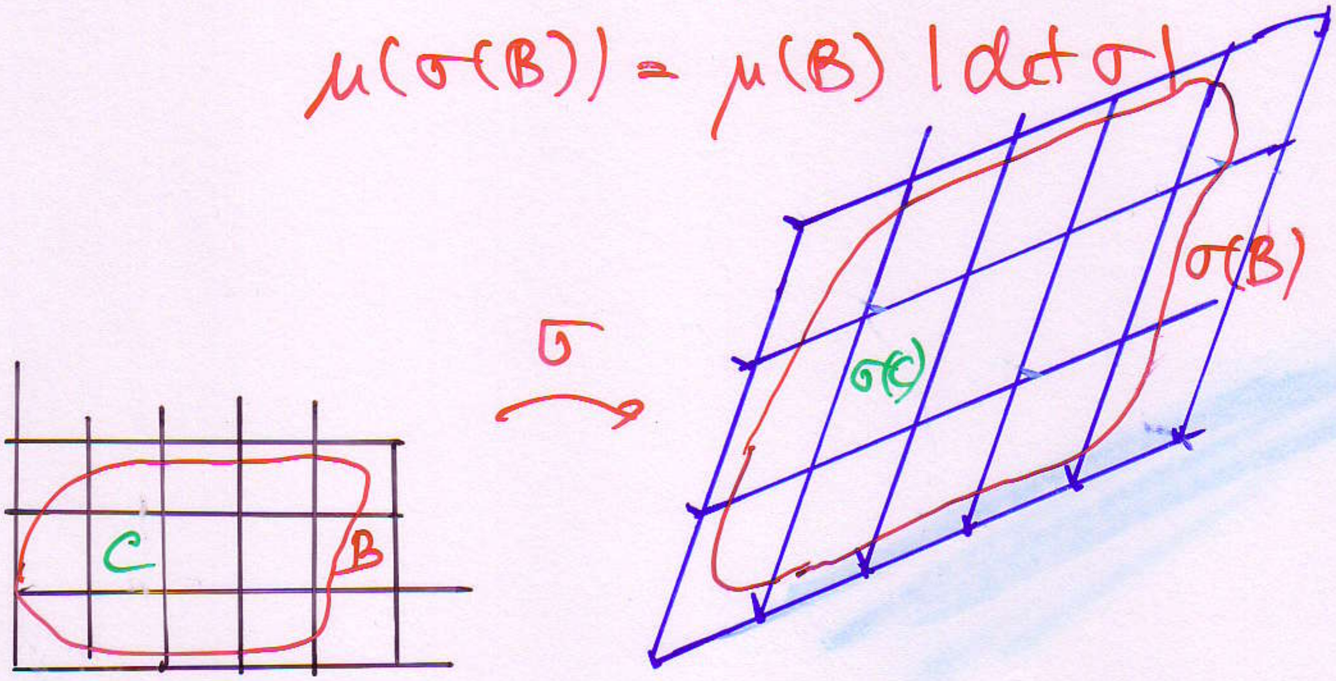
Bew $\mu(\sigma(C)) = \mu(C)$

$$\Rightarrow \underset{\displaystyle \mu(\sigma(B))}{\underset{\displaystyle \parallel}{\mu_\beta(\sigma(B))}} = \mu_\alpha(B) = \mu(B)$$

23.4.3

Satz $\sigma : \mathbb{R}^n \to \mathbb{R}^n$ affin $\Rightarrow$

$$\mu(\sigma(B)) = \mu(B)\,|\det \sigma|$$



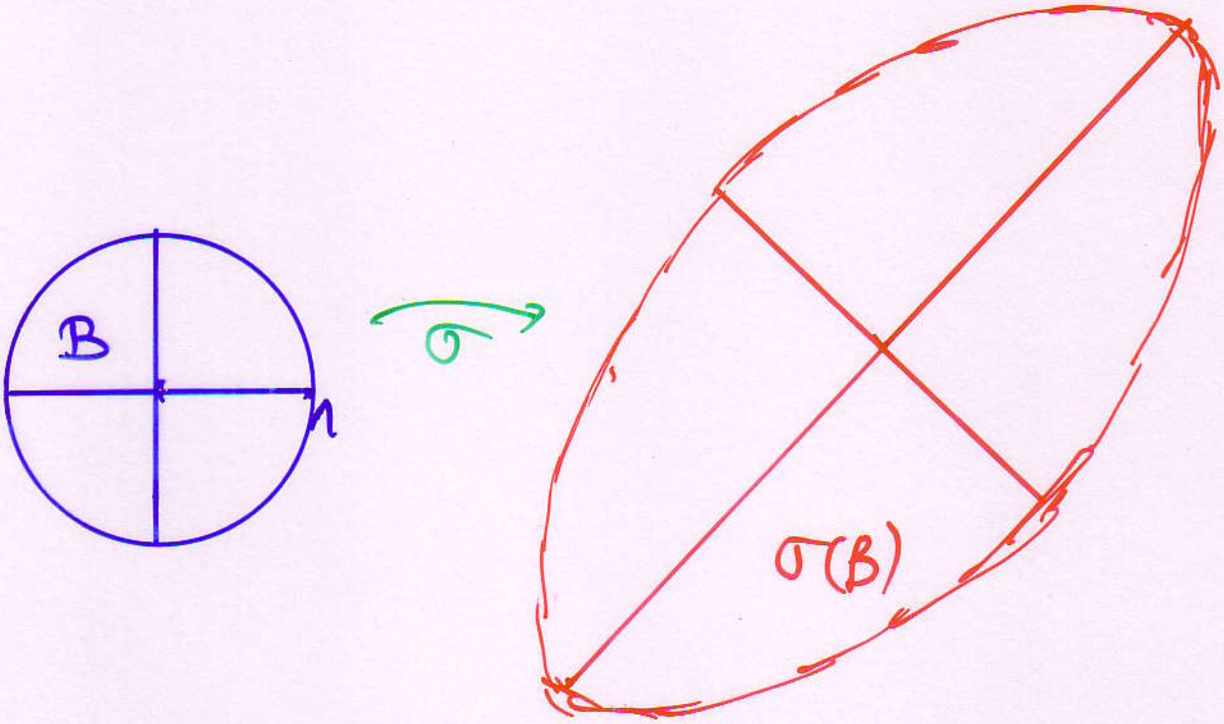
Bew C Quader, $\sigma(C)$ Spat

$$\mu(\sigma(C)) = \mu(C)\,|\det \sigma|$$

Z_n Gitter-Zerlegung von B

$\sigma(Z_n) = \{\sigma(C) \mid C \in Z_n\}$

Zerlegung von $\sigma(B)$



$$\sigma\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 3 & -3 \\ 1/2 & 1/2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

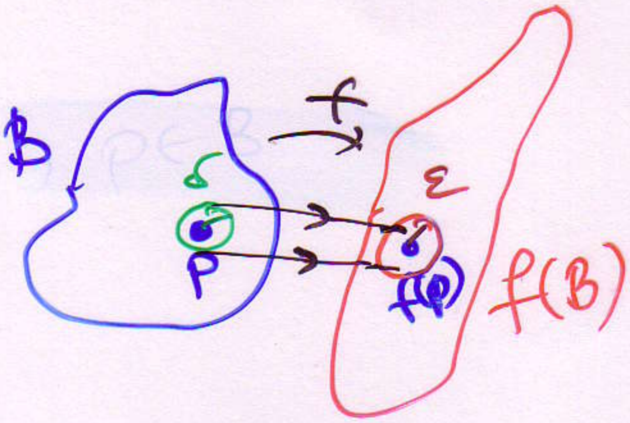
$$\det \sigma = \frac{9}{2}$$

$$\mu(\sigma(B)) = \frac{9}{2}\pi$$

23.5.1 Stetige Abb

$f : B \to \mathbb{R}^m$
$\quad\subseteq \mathbb{R}^n$

$p \in B$



f stetig an p

$\iff \forall \varepsilon > 0\ \exists \delta > 0\ \forall x \in B$
$\|x - p\| \le \delta \Rightarrow \|f(x) - f(p)\| \le \varepsilon$

$\iff \forall \underset{\in B}{x_n} \to p \quad f(x_n) \to f(p)$

$\iff f_i : B \to \mathbb{R}$ stetig $i = 1, \dots n$

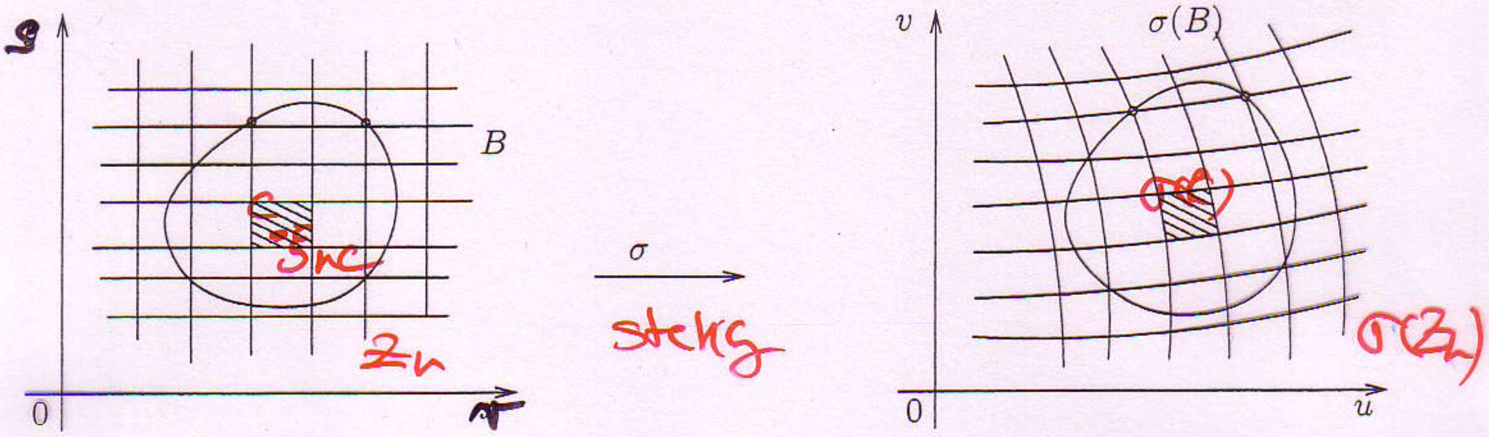
$$f(x) = \begin{pmatrix} f_1(x) \\ \vdots \\ f_n(x) \end{pmatrix}$$

f stetig auf $B \iff f$ stetig an p $\forall p \in B$

$g : C \to \mathbb{R}^k$ stetig
$f(B) \subseteq C \Rightarrow$
$g \circ f : B \to \mathbb{R}^k$ stetig

f affin $\Rightarrow f$ stetig

23.5.2 Substitution



$\tau: B \to \mathbb{R}$
stetig

$$\mu(\sigma(C)) = \tau(\xi_{nC})\,\mu(C)$$

$\boxed{\sigma, \tau \text{ Substitution}}$

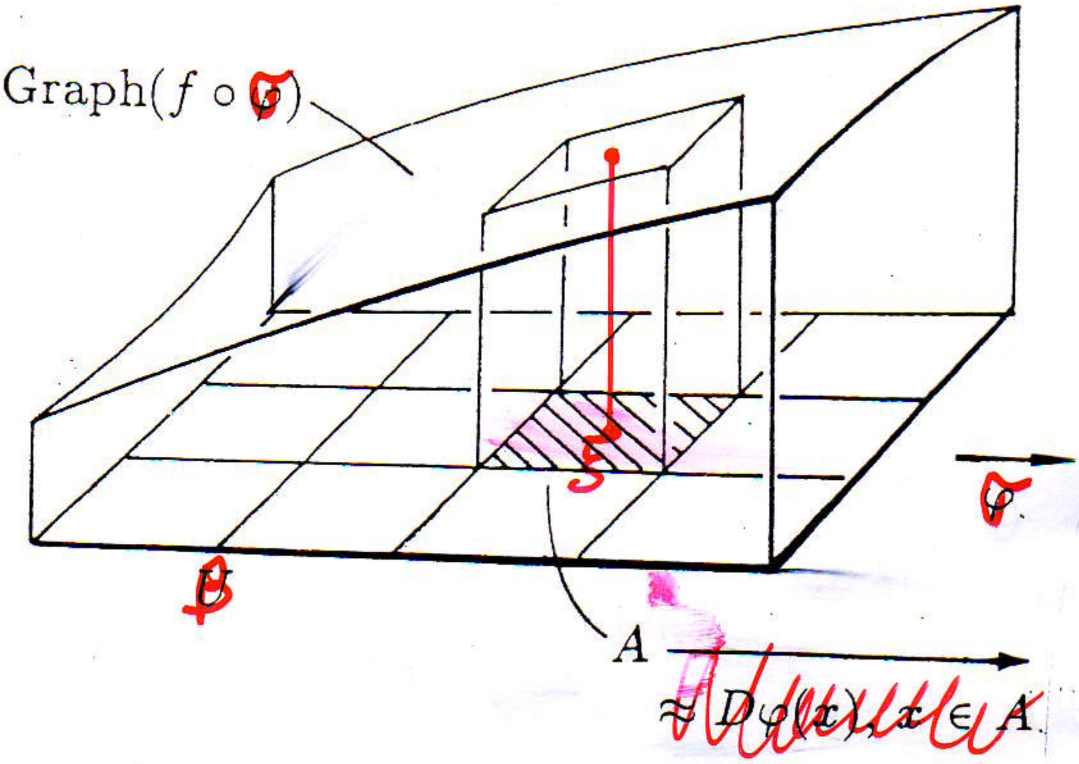
Beisp. σ affin, $\tau(\vec{r}) = \det \sigma \neq 0$

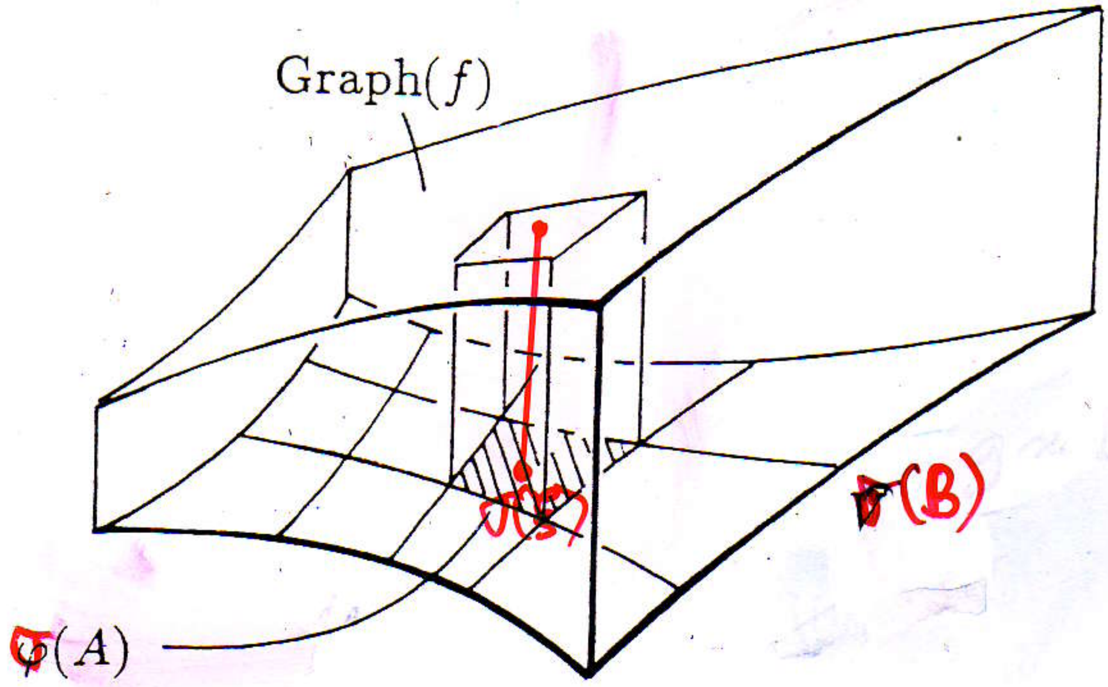
$$\int_{\sigma(B)} f(\vec{u})\, d\vec{u} = \int_B f(\sigma(\vec{r}))\cdot\tau(\vec{r})\, d\vec{r}$$

$$= \lim_{n\to\infty} \sum_{C\in Z_n} f(\sigma(\xi_{nC}))\,\tau(\xi_{nC})\,\mu(C)$$

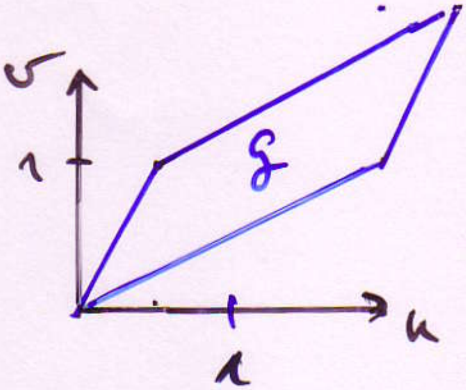
$$\lim_{n\to\infty} \sum_{D\in\sigma(Z_n)} f(\xi'_{nD})\,\mu(D)$$

mit $\xi'_{nD} = \sigma(\xi_{nC})$





$\mu(\sigma(A)) = \tau(\xi)\mu(A)$



$$f\begin{pmatrix}u\\v\end{pmatrix} = u + v^2$$

$$\mathcal{G} = \sigma(B) \qquad B = [0,1]\times[0,1]$$

$$\sigma\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}2 & 1/2\\ 1 & 1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}u\\v\end{pmatrix}$$

$$\tau\begin{pmatrix}x\\y\end{pmatrix} = \det\begin{pmatrix}2 & 1/2\\ 1 & 1\end{pmatrix} = \frac{3}{2}$$

$$\int_{\mathcal{G}} f(u,v)\, d(u,v) = \int_B f\left(\sigma\begin{pmatrix}x\\y\end{pmatrix}\right)\frac{3}{2}\, d(x,y)$$

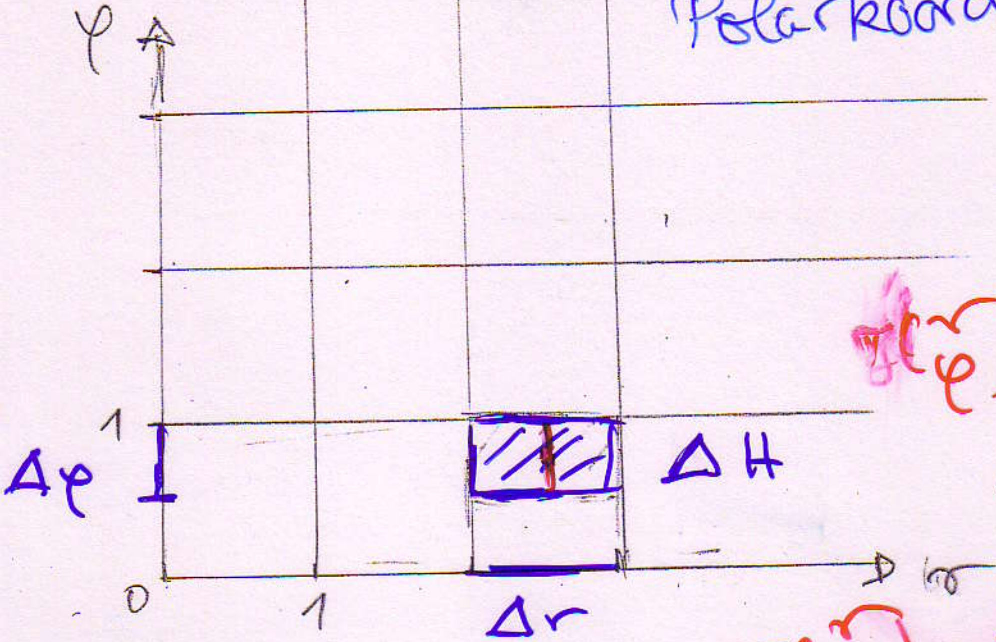
$$= \frac{3}{2}\int_0^1\int_0^1 2x + \frac{1}{2} + (x+y)^2\, dx\, dy$$

$$= \frac{3}{2}\int_0^1 \frac{3}{2} + \frac{1}{3}(x+y)^3\Big|_{x=0}^{x=1}\, dy$$

$$= \frac{3}{2}\left(\frac{3}{2} - \frac{1}{12} + \frac{16}{12}\right) = \frac{33}{8}$$

23.6.1 Polarkoordinaten

B

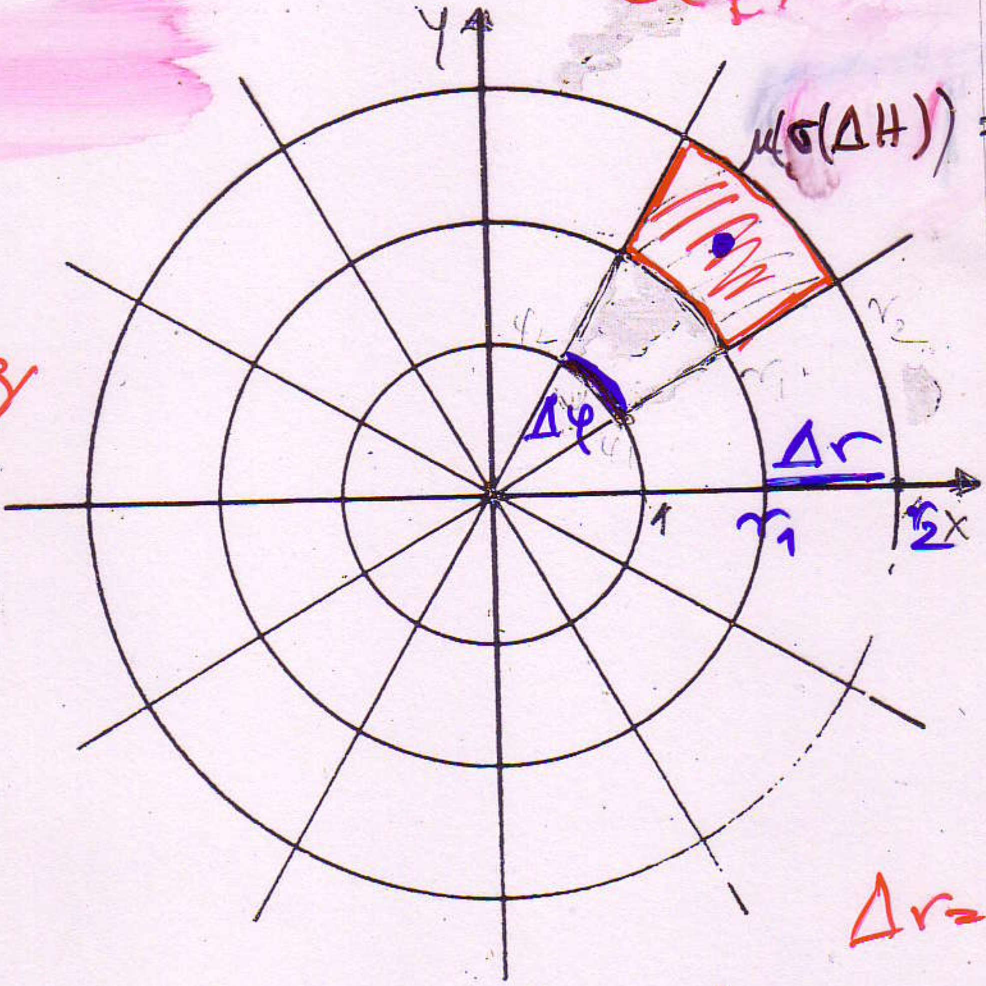


$$\sigma\binom{r}{\varphi} = r\begin{pmatrix}\cos\varphi\\ \sin\varphi\end{pmatrix} = \sigma\binom{r}{\varphi}$$

$$\tau\binom{r}{\varphi} = r$$

$$\mu(\sigma(\Delta H)) = r\,\mu(\Delta H)$$

$\sigma(B)$



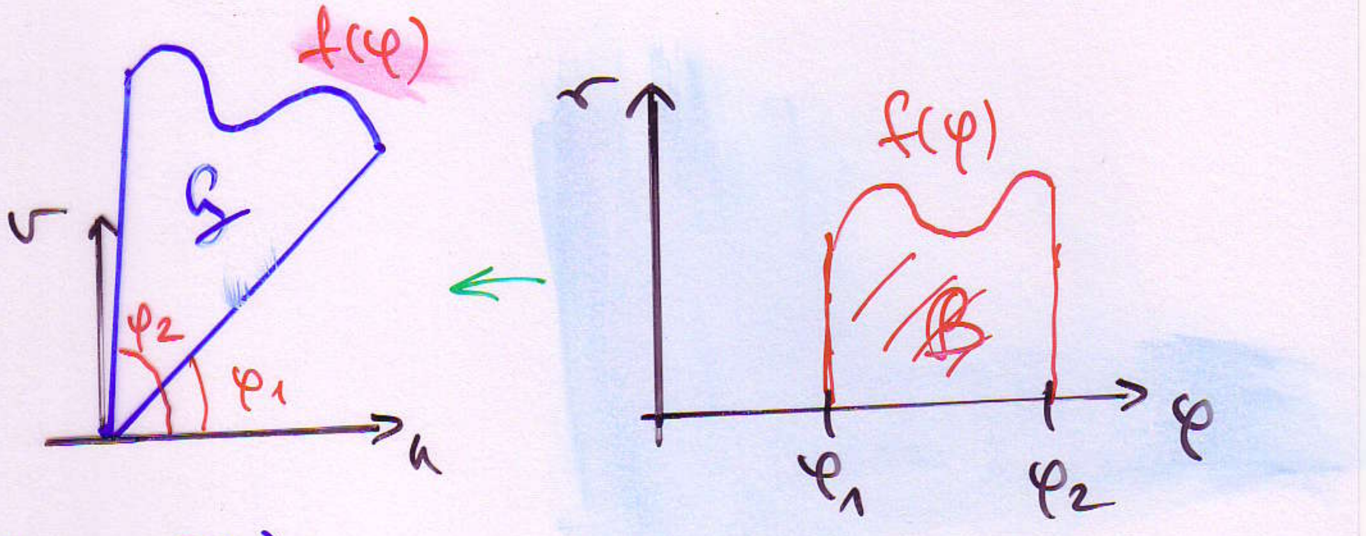
$$\Delta r = r_2 - r_1$$

$$\mu(\sigma(\Delta H)) = \frac{1}{2}\Delta\varphi\, r_2^2 - \frac{1}{2}\Delta\varphi\, r_1^2$$

$$= r\,\Delta\varphi\,\Delta r \qquad r = \frac{1}{2}(r_1 + r_2)$$

3. Binomi

$$\mathcal{G} = \left\{ \binom{r}{\varphi} \;\middle|\; \varphi_1 \le \varphi \le \varphi_2 ,\; 0 \le r \le f(\varphi) \right\}$$



$$\mathcal{G} = \sigma(B)$$

$$\mu(\sigma(B)) = \int_{\sigma(B)} 1 \, d(u,v)$$

$$= \int_B r \, d(r,\varphi)$$

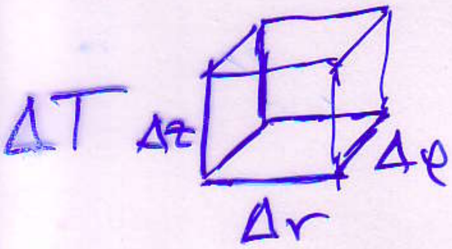
$$= \int_{\varphi_1}^{\varphi_2} \int_0^{f(\varphi)} r \, dr \, d\varphi$$

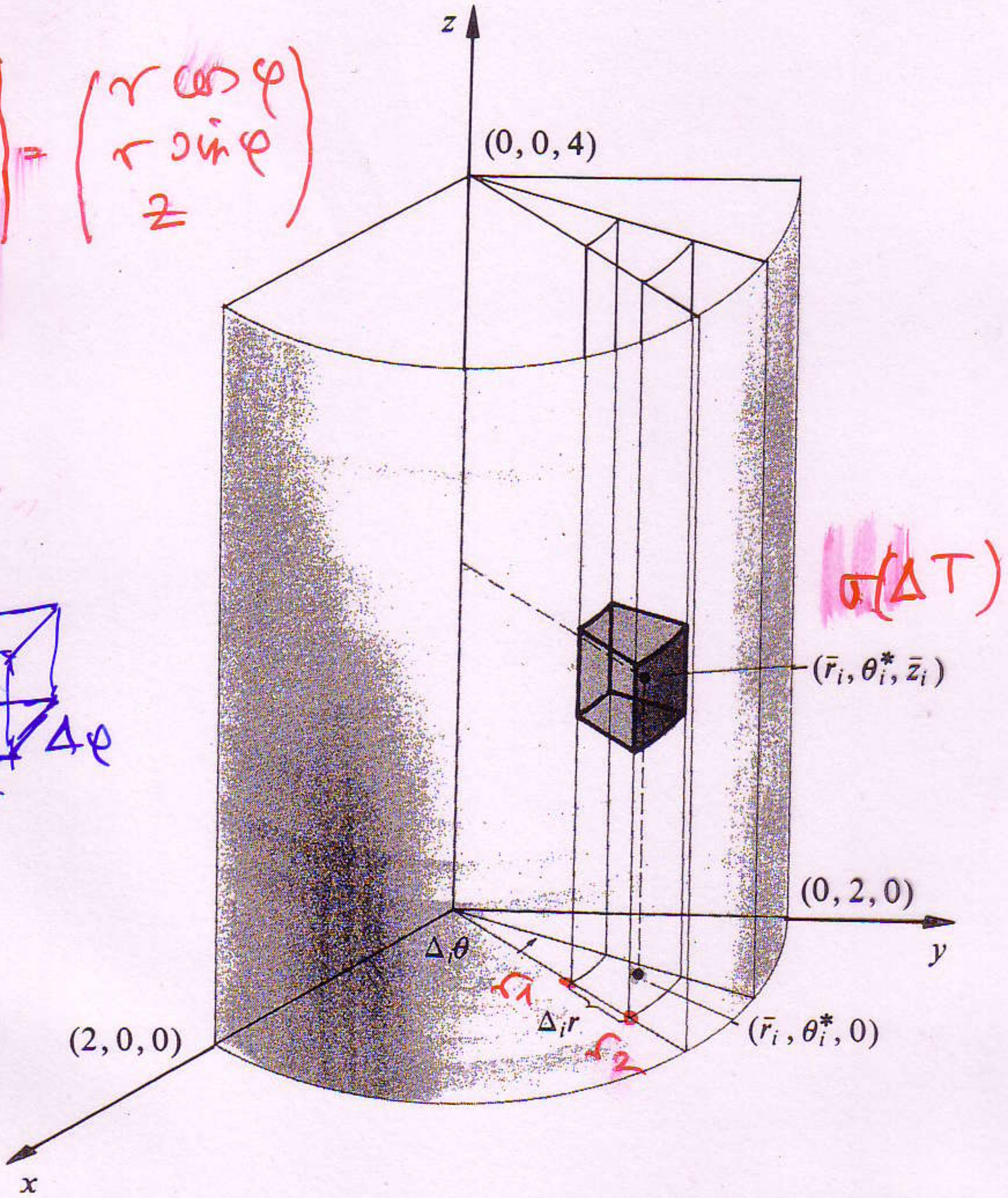
$$= \int_{\varphi_1}^{\varphi_2} \frac{f^2}{2}(\varphi) \, d\varphi \qquad f(\varphi) = \sqrt{\varphi}$$

$$= \int_{\varphi_1}^{\varphi_2} \frac{\varphi}{2} \, d\varphi = \frac{1}{4}(\varphi_2^2 - \varphi_1^2)$$

23.6.4 Zylinderkoordinaten

$$\sigma\begin{pmatrix} r \\ \varphi \\ z \end{pmatrix} = \begin{pmatrix} r\cos\varphi \\ r\sin\varphi \\ z \end{pmatrix}$$

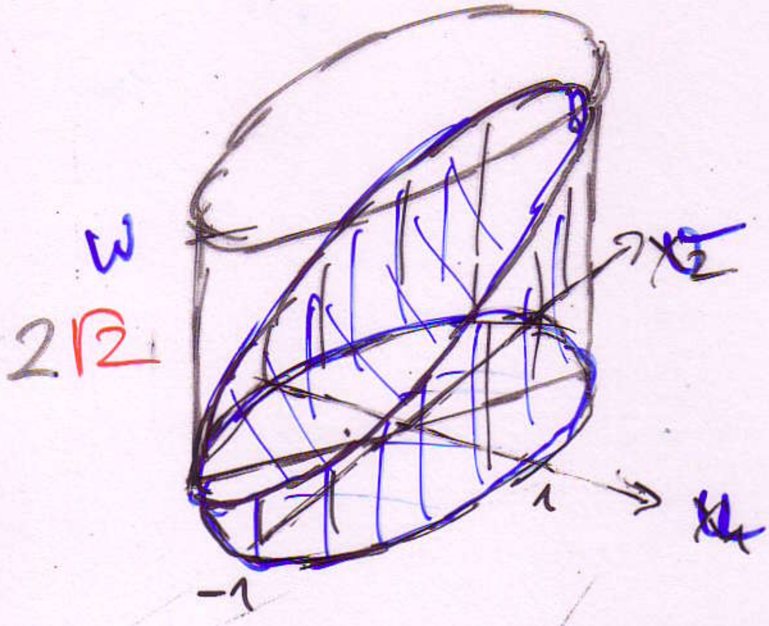




$$\mu(\sigma(\Delta T)) = \bar{r}\,\Delta\varphi\,\Delta r\,\Delta z$$

$$\bar{r} = \frac{1}{2}(r_1 + r_2)$$

$$\tau\begin{pmatrix} r \\ \varphi \\ z \end{pmatrix} = r$$



$$G = \left\{ \begin{pmatrix} u \\ v \\ w \end{pmatrix} \;\middle|\; u^2+v^2 \le 1,\ 0 \le w \le u+v+\sqrt{2} \right\}$$

$$G = \sigma(B) \qquad \sigma\begin{pmatrix} r \\ \varphi \\ z \end{pmatrix} = \begin{pmatrix} r\cos\varphi \\ r\sin\varphi \\ z \end{pmatrix} \qquad \tau\begin{pmatrix} r \\ \varphi \\ z \end{pmatrix} = r$$

$$B = \left\{ \begin{pmatrix} r \\ \varphi \\ z \end{pmatrix} \;\middle|\; \begin{array}{l} 0 \le r \le 1 \\ 0 \le \varphi \le 2\pi \\ 0 \le z \le r\cos\varphi + r\sin\varphi + \sqrt{2} \end{array} \right.$$

$$\mu(G) = \int_G 1 \, d(u,v,w) = \int_B r \, d(r,\varphi,z)$$

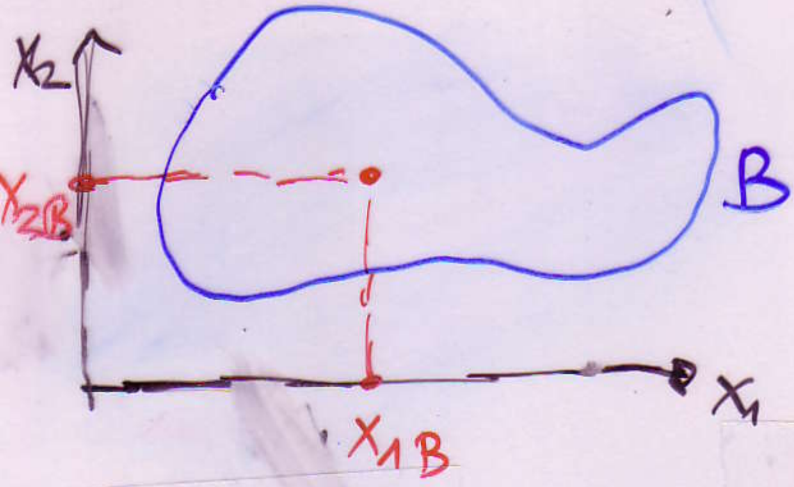
$$= \int_0^1 \int_0^{2\pi} \int_0^{r\cos\varphi + r\sin\varphi + \sqrt{2}} 1 \cdot r \, dz\, d\varphi\, dr$$

$$= \int_0^1 \int_0^{2\pi} r^2(\cos\varphi + \sin\varphi) + r\sqrt{2} \, d\varphi\, dr$$

$$= \int_0^1 r^2 \underbrace{[\sin\varphi - \cos\varphi]_0^{2\pi}}_{0} + 2\pi r\sqrt{2} \, dr$$

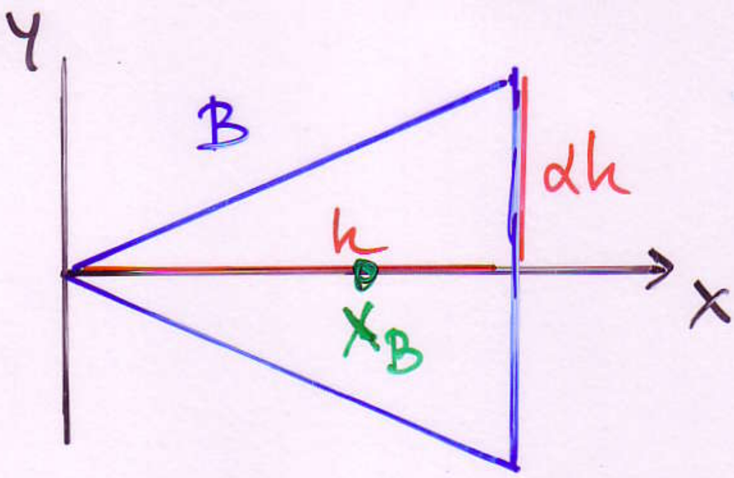
$$= \pi r^2 \sqrt{2}$$

23.6.5 Schwerpunkt



$$x_{i_B} = \frac{1}{\mu(B)} \int_B x_i \, d\vec{x}$$

Beispiel



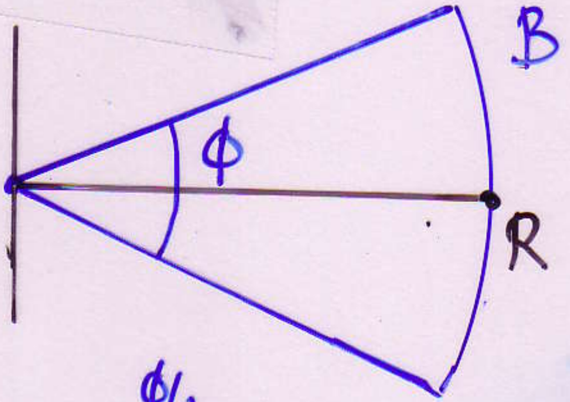
$$x_B = \frac{1}{\alpha h^2} \int_B x \, dx$$

$$= \frac{1}{\alpha h^2} \int_0^h \int_{-\alpha x}^{\alpha x} x \, dy \, dx$$

$$= \frac{1}{\alpha h^2} \int_0^h 2\alpha x^2 \, dx = \frac{1}{\alpha h^2} \frac{2}{3} \alpha h^3$$

$$= \frac{2}{3} h$$

21



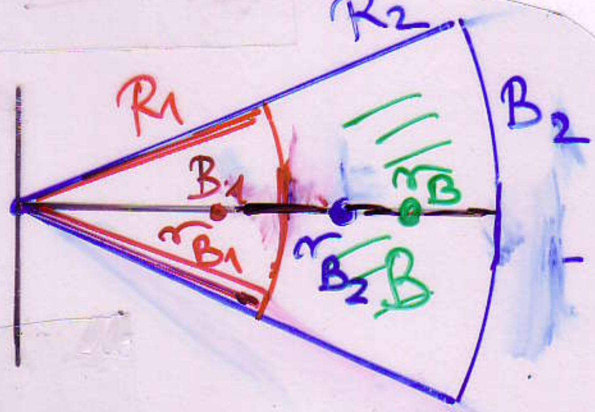
$$\int_B x \, d(x,y)$$

$$= \int_{-\phi/2}^{\phi/2} \int_0^R r\cos\varphi \; r \, dr \, d\varphi$$

$$= \int_{-\phi/2}^{\phi/2} \frac{1}{3} R^3 \cos\varphi \, d\varphi = \frac{1}{3} R^3 \sin\varphi \Big|_{-\phi/2}^{\phi/2} = \frac{2}{3} R^3 \sin \phi/2$$

$$\mu(B) = \frac{1}{2} R^2 \phi$$

$$x_B = r_B = \underbrace{\frac{4}{3} R \frac{1}{\phi} \sin \phi/2}_{\to \frac{2}{3} R}$$



Hebel

$$(r_{B_2} - r_{B_1})\,\mu(B_1)$$
$$= (r_B - r_{B_2})\,\mu(B)$$

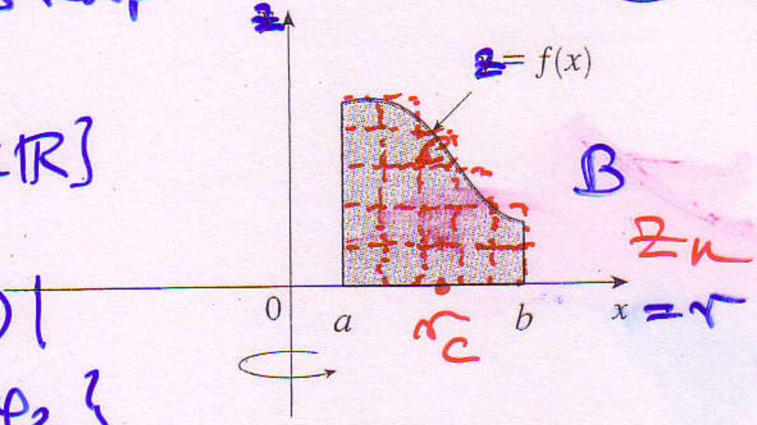
$$r_B = \frac{\mu(B_1)}{\mu(B)} (r_{B_2} - r_{B_1}) + r_{B_2}$$

$$= \frac{R_1^2}{R_2^2 - R_1^2} \frac{4}{3} \frac{1}{\phi} \sin\frac{\phi}{2} (R_2 - R_1) + \frac{4}{3} R_2 \frac{1}{\phi} \sin\frac{\phi}{2}$$

$$= \frac{4}{3} \frac{1}{\phi} \sin\frac{\phi}{2} \left(\frac{R_1^2}{R_2 + R_1} + R_2 \right) \geq R_1$$

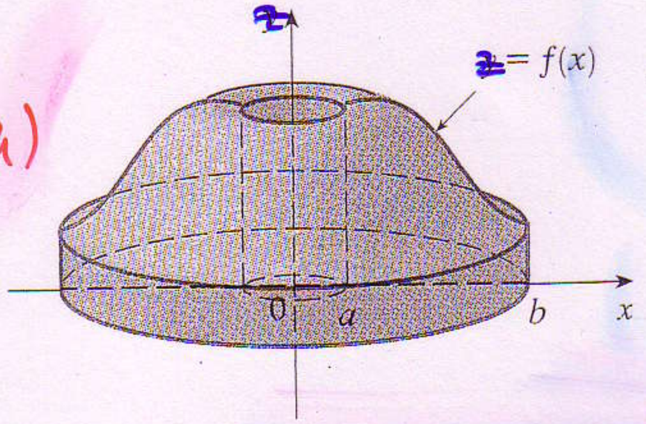
für $\phi \to 0$

23.66 Rotationskörper



$B \subseteq \{(r,z) \mid r \geq 0, z \in \mathbb{R}\}$

$D = \{(r\cos\varphi, r\sin\varphi, z) \mid$
$(r,z) \in B, \varphi_1 \leq \varphi \leq \varphi_2\}$



$\mu(D) = r_B \cdot \mu(B)(\varphi_2 - \varphi_1)$

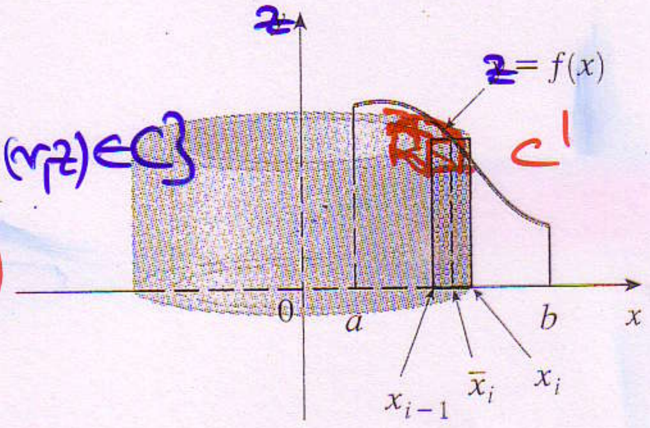
Bew

$r_B = \frac{1}{\mu(B)} \int_B r \, d(r,z)$

FIGURE 3

$= \frac{1}{\mu(B)} \lim_{n \to \infty} r_C \, \mu(C)$

$\varphi_1 = 0, \varphi_2 = \frac{\pi}{4}$

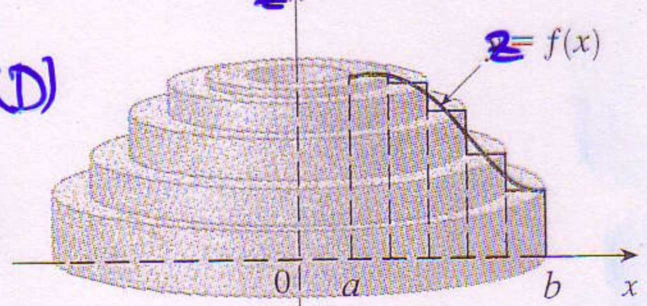


$C' = \{(r\cos\varphi, r\sin\varphi, z) \mid (r,z) \in C\}$

$\mu(C') = r_C \, \mu(C)(\varphi_2 - \varphi_1)$

⇝ Zerlegung von D

$O(Z_n', D) - U(Z_n', D) \to 0$



$(\varphi_2 - \varphi_1) \sum r_C \, \mu(C) \to \mu(D)$

$\to (\varphi_2 - \varphi_1) \, r_B \, \mu(B)$

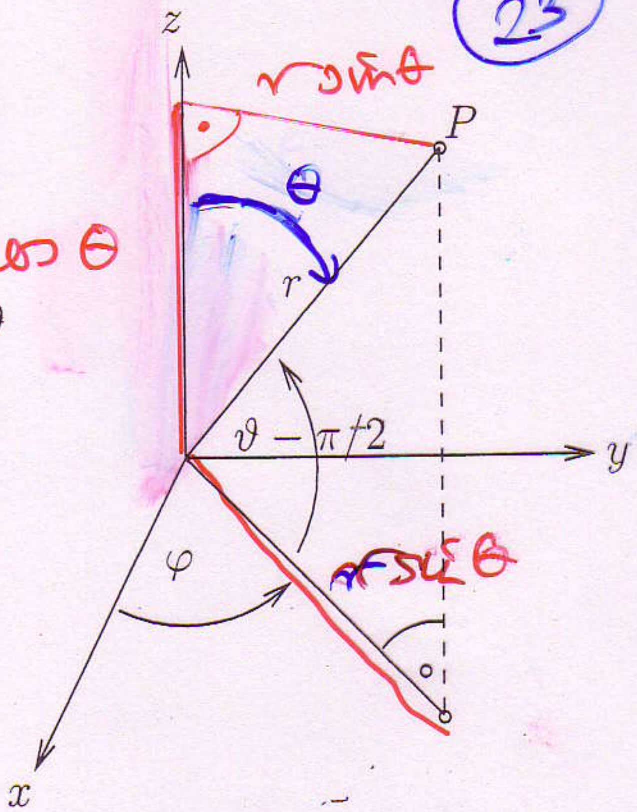
Z_n' nicht geeignet für $\int$ da Weite $\not\to 0$

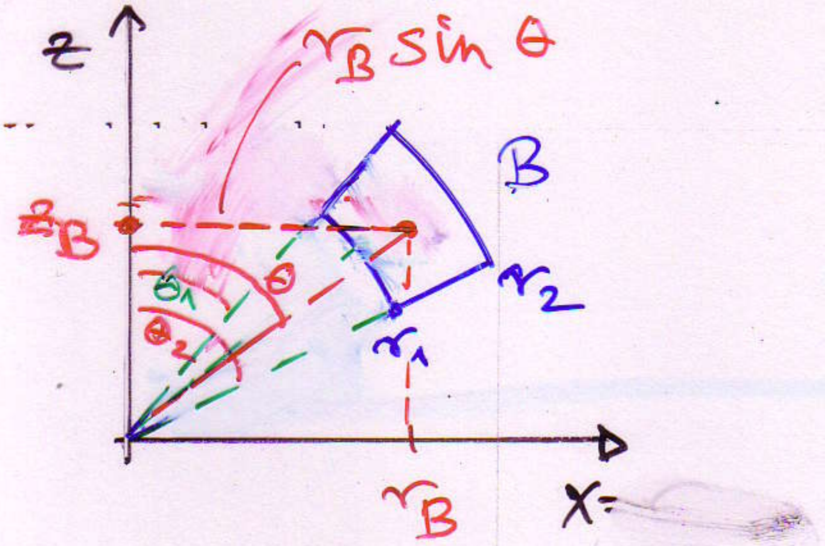
Kugelkoordinaten Die Kugelkoordinaten (r, ϑ, φ) sind mit den kartesischen Koordinaten (x, y, z) verknüpft durch

$$x = r\cos\varphi\,\sin\vartheta,\ \ y = r\sin\varphi\,\sin\vartheta,\ \ z = r\cos\vartheta$$

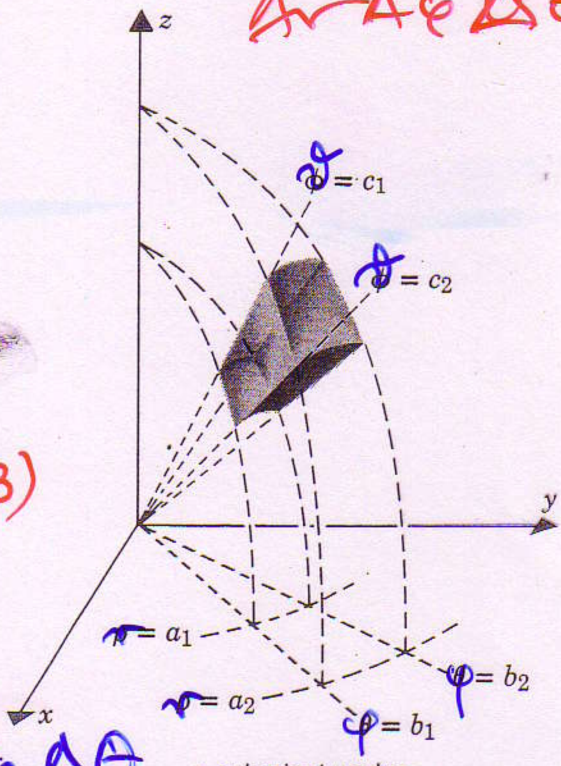
wobei $r \geq 0$, $0 \leq \vartheta \leq \pi$ und $0 \leq \varphi \leq 2\pi$. Die Substitutionsfunktion ist

$$(x, y, z) = \sigma(r, \varphi, \vartheta)$$
$$(r\cos\varphi\,\sin\vartheta,\ r\sin\varphi\,\sin\vartheta,\ r\cos\vartheta)$$





$\Delta r\,\Delta\varphi\,\Delta\theta$



a spherical wedge

$$\mu(D) = (r_B \sin\theta_B\,\Delta\varphi)\,\mu(B)$$

$$\mu(B) = \frac{r_1 + r_2}{2}\,\Delta r\,\Delta\theta$$

$$\mu(D) = \bar{r}^2 \sin\theta_B\,\Delta r\,\Delta\varphi\,\Delta\theta$$

$$\bar{r} = \sqrt{r_B \cdot \frac{r_1 + r_2}{2}}$$

$$\boxed{\mathcal{E}(r, \varphi, \theta) = r^2 \sin\theta}$$

G = Kugel um 0
Radius R

σ Kugelkoordinaten

$G = \sigma(B) \quad B = [0,R] \times [0,2\pi] \times [0,\pi]$

$\tau = r^2 \sin\theta$

$$\mu(G) = \int_G 1$$

$$= \int_B r^2 \sin\theta \; d(r,\varphi,\theta)$$

$$= \int_0^R \int_0^{2\pi} \int_0^{\pi} r^2 \sin\theta \; d\theta \, d\varphi \, dr$$

$$= \int_0^R \int_0^{2\pi} \left[-r^2 \cos\theta\right]_0^{\pi} d\varphi \, dr$$

$$= \int_0^R \int_0^{2\pi} 2r^2 \, d\varphi \, dr$$

$$= \int_0^R 4r^2\pi \, dr$$

$$= \frac{4}{3} R^3 \pi$$