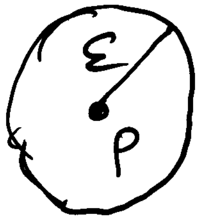
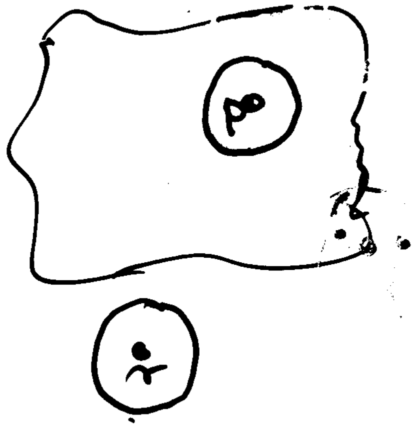
1.9.1.3

$U_\varepsilon(p) = \{x \in \mathbb{R}^n \mid \|x-p\| < \varepsilon\}$



ε-Umgebung von p

M



p innerer Punkt
M Umgebung von p
q Randpunkt

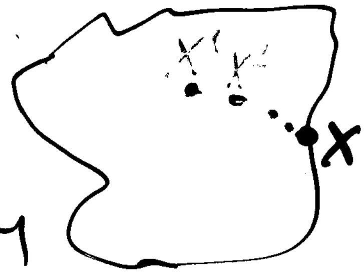
r isolierter Punkt $\exists \varepsilon$

p, q Häufungspunkte

M offen: $\forall p \in M$ innere Pkte

M abgeschlossen:
$\delta(M) = \{q \mid q \text{ Randpkt}\} \subseteq M$

$\Longleftrightarrow \underset{\in M}{x^{(k)}} \to x \to x \in M$



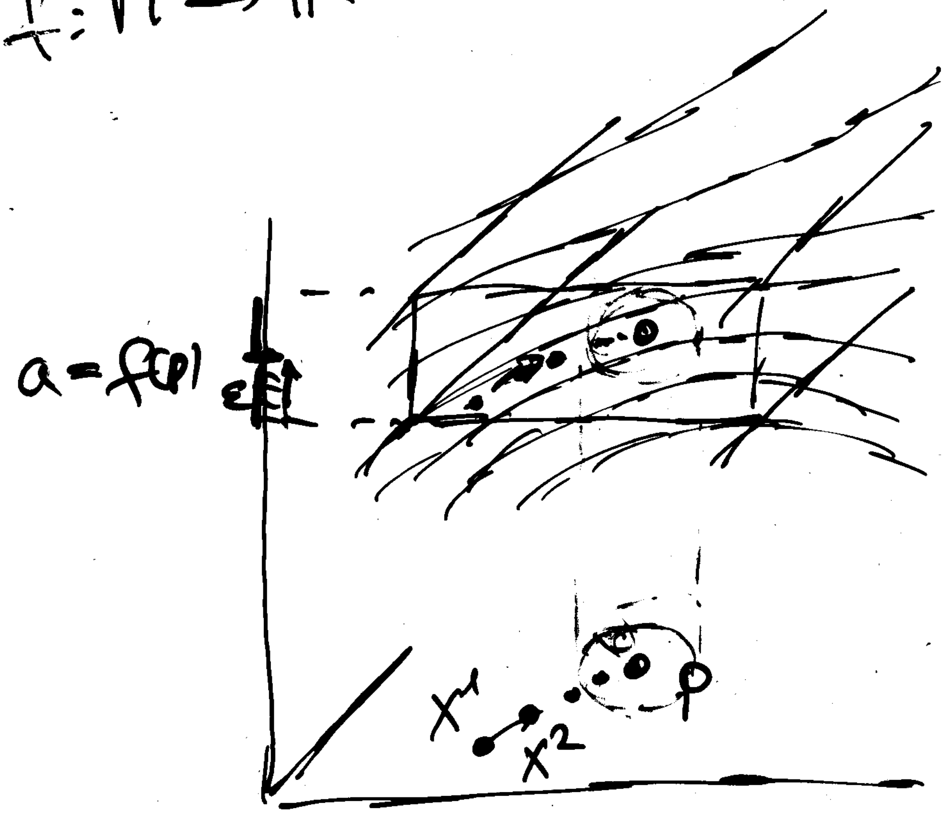
M

M beschränkt $\Longleftrightarrow \exists R \quad M \subseteq U_R(0)$

19.1.4 M kompakt $\Longleftrightarrow$ abgeschlossen + beschränkt
$\Longleftrightarrow$ für alle U_i $(i \in I)$, U_i offen gilt
$M \subseteq \bigcup_{i \in I} U_i \Rightarrow \exists I' \subseteq I$ endlich
$M \subseteq \bigcup_{i \in I'} U_i$

19.1.6/7

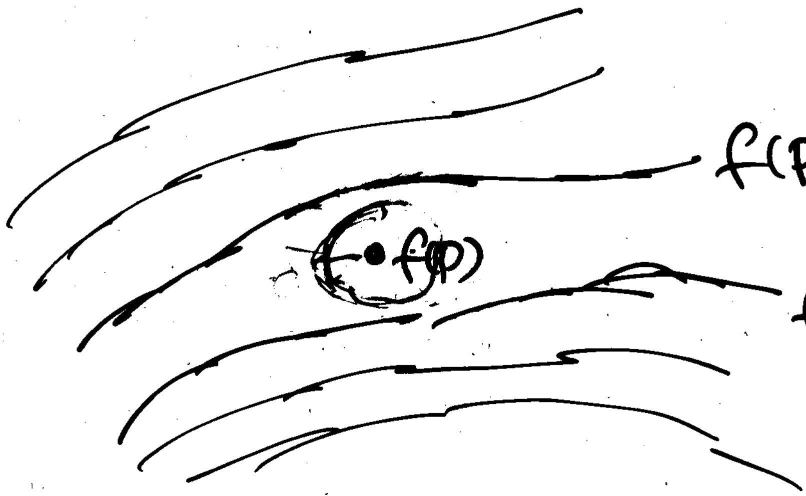
$f: M \to \mathbb{R}$



$f(x^k) \to a$

für $x^k \to p$

$\forall \varepsilon > 0$



$p \in M$

f an p stetig $\Longleftrightarrow$

$\forall \varepsilon > 0 \; \exists \delta > 0 \; \forall x \in M \; \|x-p\| < \delta \Rightarrow |f(x) - f(p)| < \varepsilon$

$\Longleftrightarrow \; \forall \underset{\in M}{x^k} \to p \quad f(x^k) \to f(p)$

f, g stetig an p

$\Rightarrow f+g,\ f \cdot g,\ \frac{f}{g}$ stetig an p

$f: M \to \mathbb{R}$ stetig $\Leftrightarrow$

stetig an allen $p \in M$

M kompakt $\Rightarrow \exists\, x_{max}, x_{min} \in M$

$\forall x \in M \quad f(x_{min}) \leq f(x) \leq f(x_{max})$

und

$f: M \to \mathbb{R}$ gleichmäßig stetig

$\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall x, p \in M$

$\|x - p\| < \delta \Rightarrow |f(x) - f(p)| < \varepsilon$

1.9.2.1.

$f: \mathbb{R}^n \to \mathbb{R}$ homogen linear

$\iff f(\vec{x}+\vec{y}) = f(\vec{x}) + f(\vec{y})$

$f(\lambda \vec{x}) = \lambda f(\vec{x})$

$\iff \exists a_1, .., a_n \in \mathbb{R}$

$f(x) = a_1 x_1 + .. + a_n x_n$

$g: \mathbb{R}^n \to \mathbb{R}$ affin linear

$\iff \exists f: \mathbb{R}^n \to \mathbb{R}$ homogen linear

$\exists b \in \mathbb{R}$

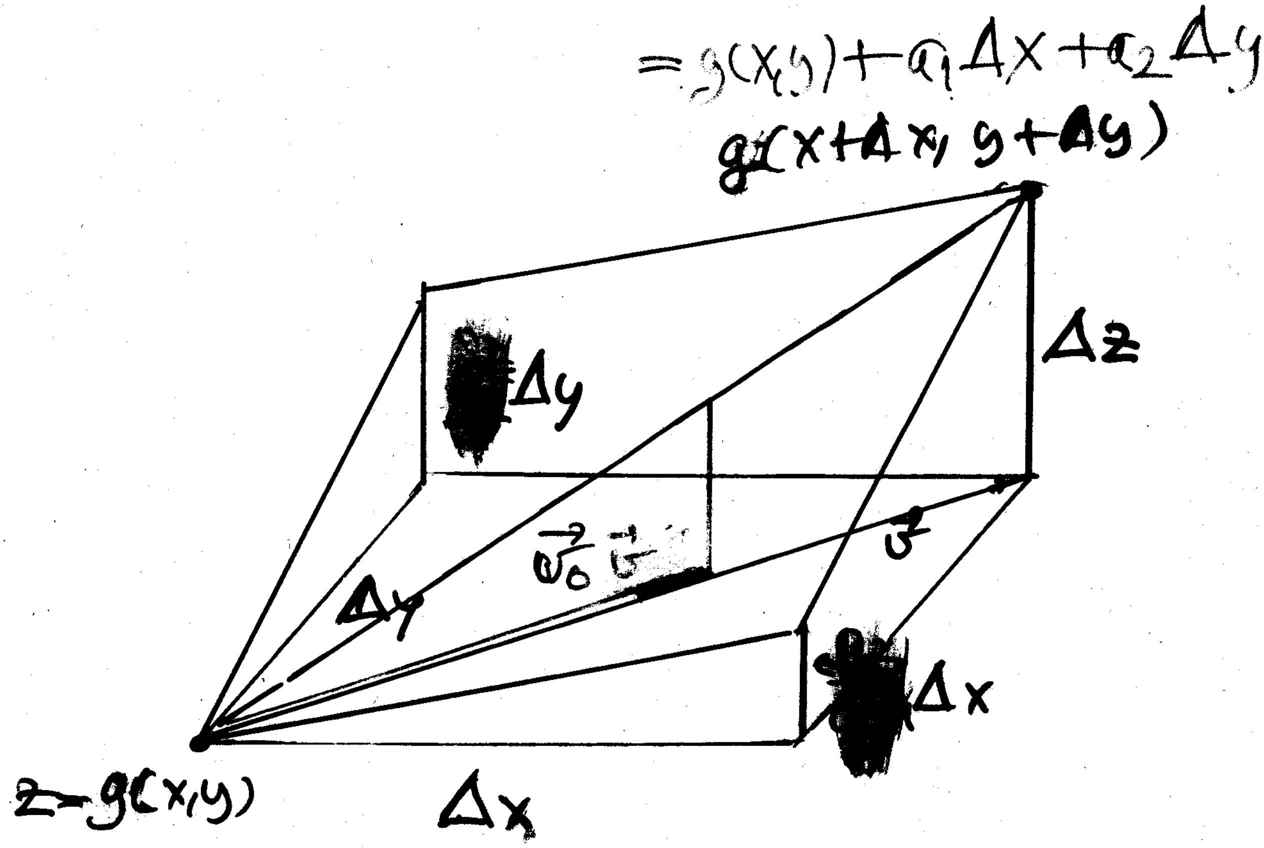
$g(x) = f(\vec{x}) + b$

$\iff \exists a_1, .., a_n, b \in \mathbb{R}$

$g(x) = a_1 x_1 + .. + a_n x_n + b$

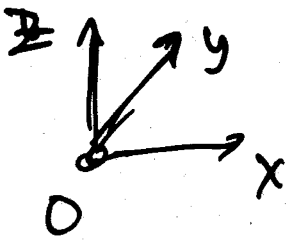
19.2.1.

$g(x,y) = a_1 x + a_1 y + b$



$p = (x,y) \quad \vec{v} = (\Delta x, \Delta y)$

$$d_{\vec{v}}\, g(p) = \frac{g(\vec{v}+p) - g(p)}{\|\vec{v}\|}$$



$$\vec{v}_0 = \frac{\vec{v}}{\|\vec{v}_0\|} \qquad g(\vec{v}_0 + p) - g(p)$$

$$= a_1 v_{01} + a_2 v_{02}$$

$$= \langle \operatorname{grad} g \mid \vec{v}_0 \rangle$$

$\operatorname{grad} g \mathrel{\hat{=}} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$

maximal für $\vec{v}_0 = \lambda \operatorname{grad} g$

$\lambda > 0$

1.9.2.1.

$f\binom{x_1}{x_2} = a_1 x_1 + a_2 x_2$

$\vec{g} \hat{=} \operatorname{grad} f = (a_1, a_2)$

$\vec{g}_0 = \frac{1}{\|\vec{g}\|}\vec{g}$

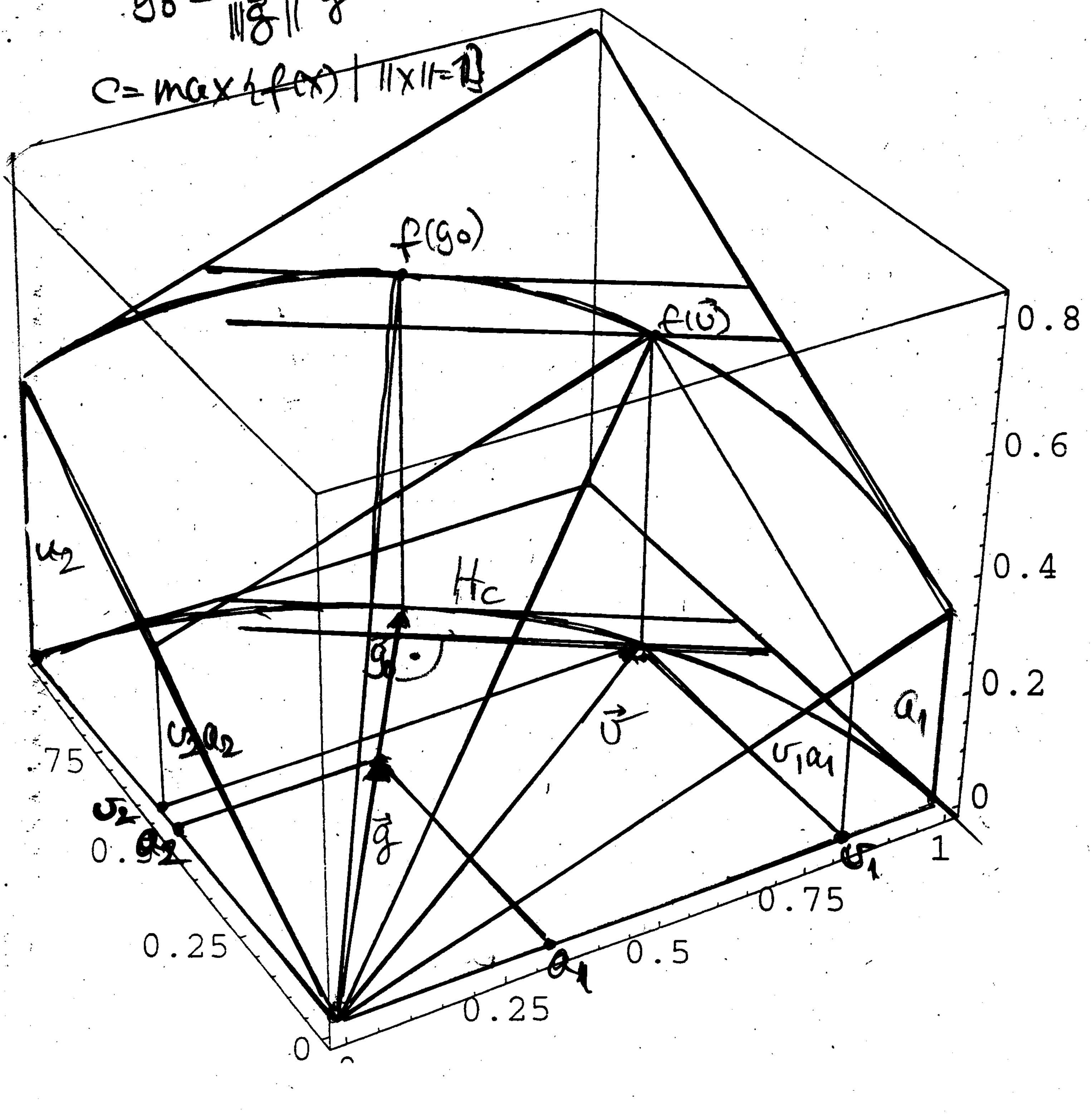
$c = \max\{f(x) \mid \|x\| = 1\}$

$D_v f(0) =$

$\frac{d}{dt} f(t\vec{v}) = f(v)$

$= v_1 a_1 + v_2 a_2$.

$= \langle \vec{g} \mid \vec{v} \rangle$



In[1}]

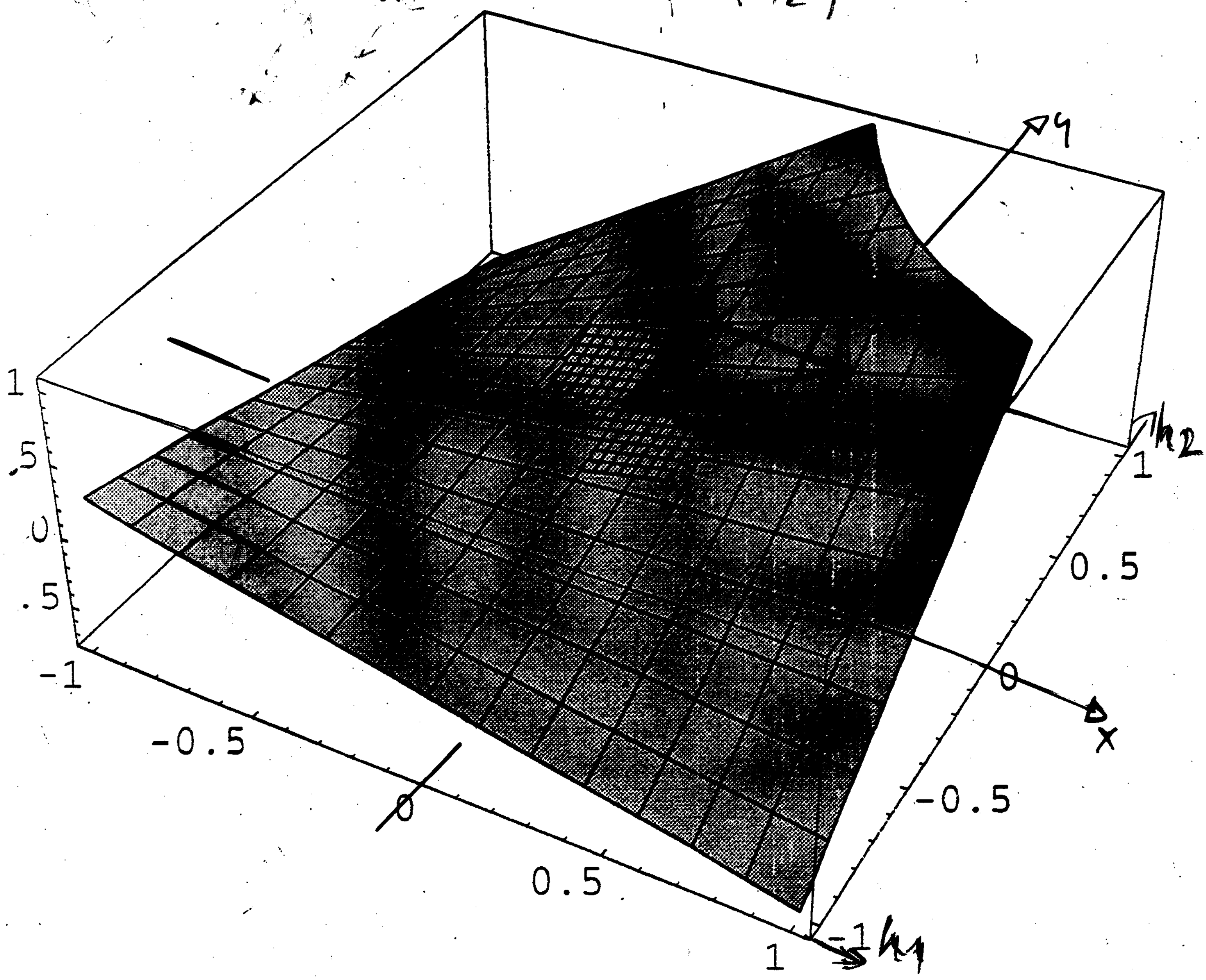
1.9.2.2

$f: \mathbb{R}^2 \to \mathbb{R} \qquad f\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_1 x_2 \qquad \frac{R(\vec{h})}{\|h\|} \to 0$

$$f(p+h) = f(p) + \underbrace{p_2 h_1 + p_1 h_2}_{df(p)(\vec{h})} + \underbrace{h_1 h_2}_{R(\vec{h})}$$

$\operatorname{grad} f(p) = \begin{pmatrix} p_2 \\ p_1 \end{pmatrix}$

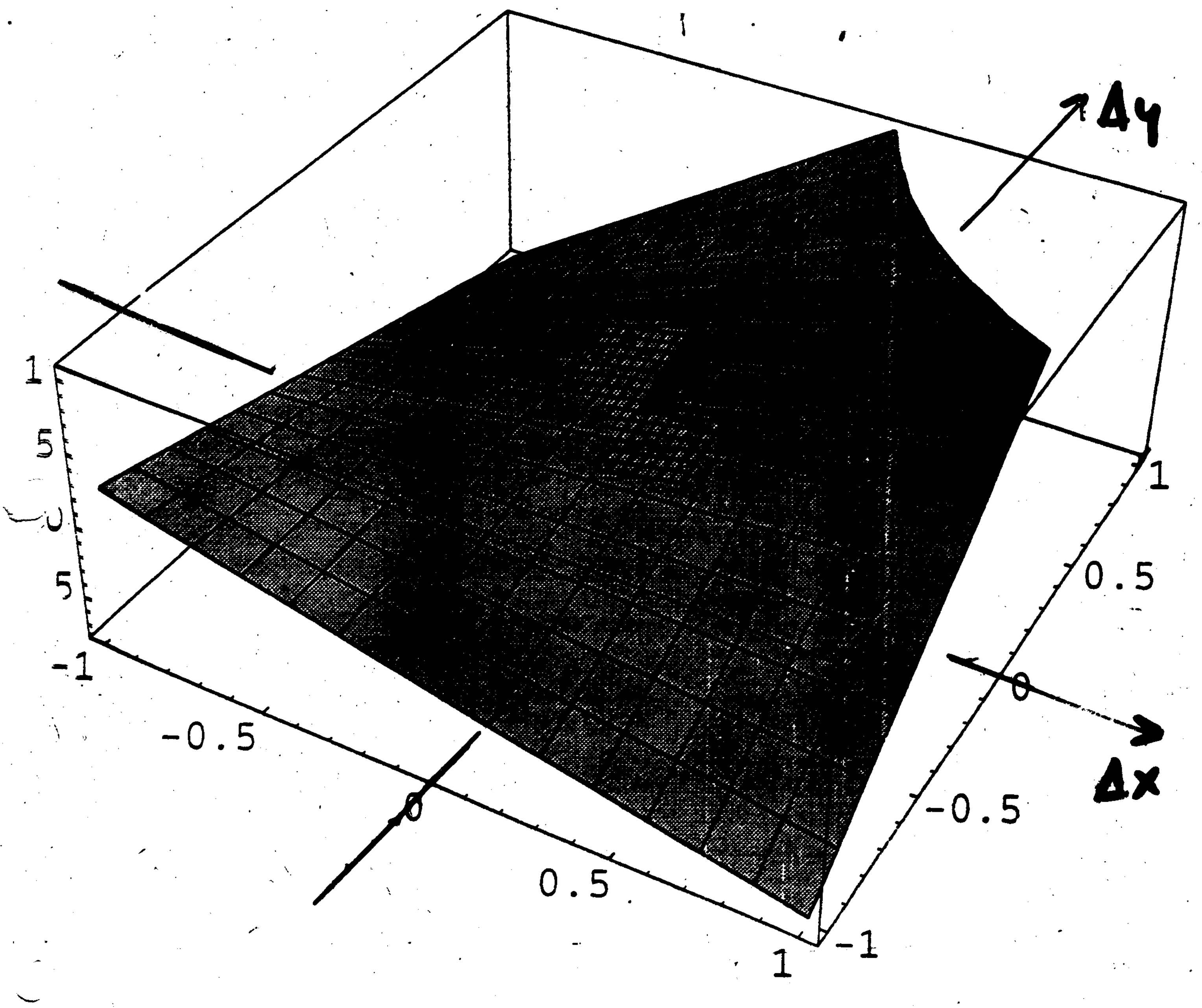
$p = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$




```
Plot3D[(1/2+h1)(1/2+h2), {h1,-1,1} ,{h2,-1,1} ]

Plot3D[1/4+ {1/2,1/2}.{h1,h2}, {h1,-1/4,1/4} ,{h2,-1/4,1/4} ]
```

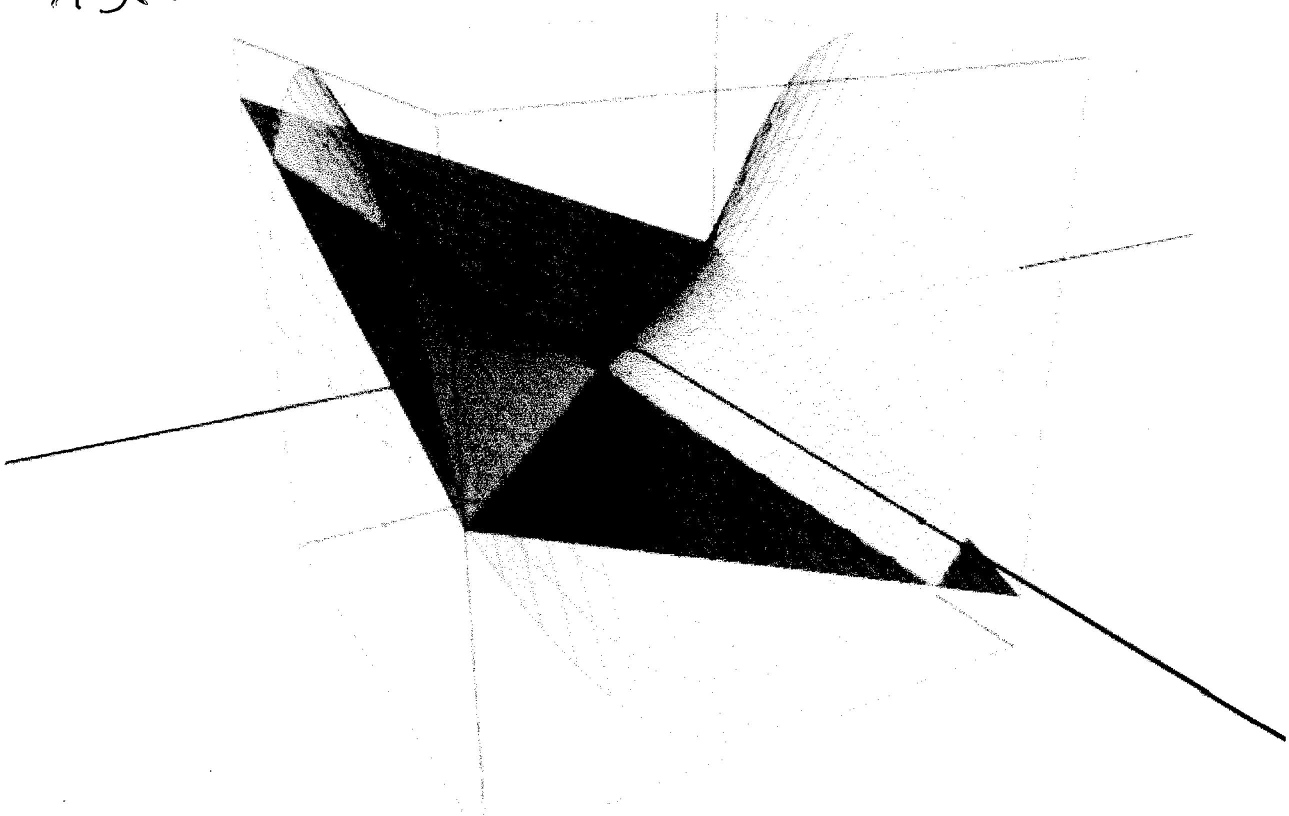
19.2.2

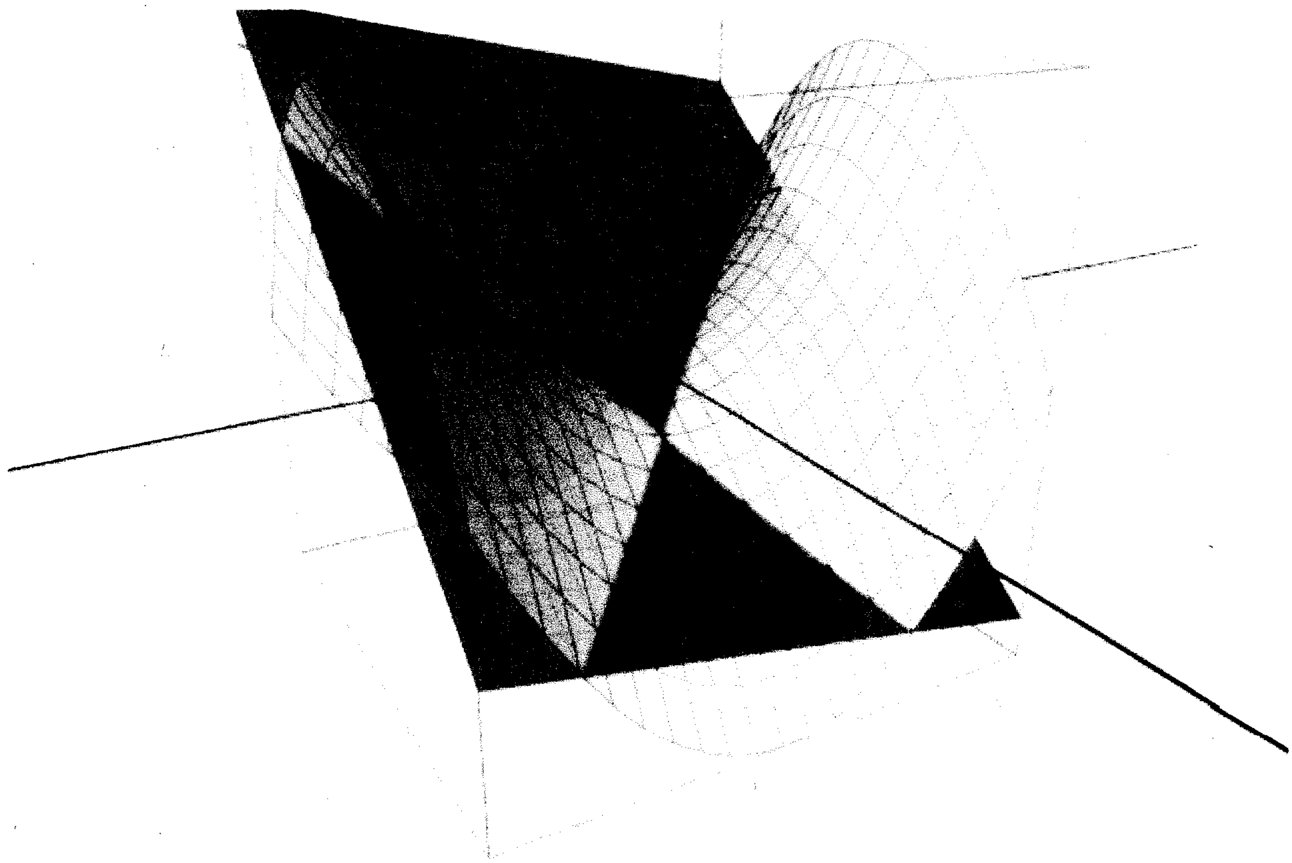


$$z = f(x,y) = xy$$

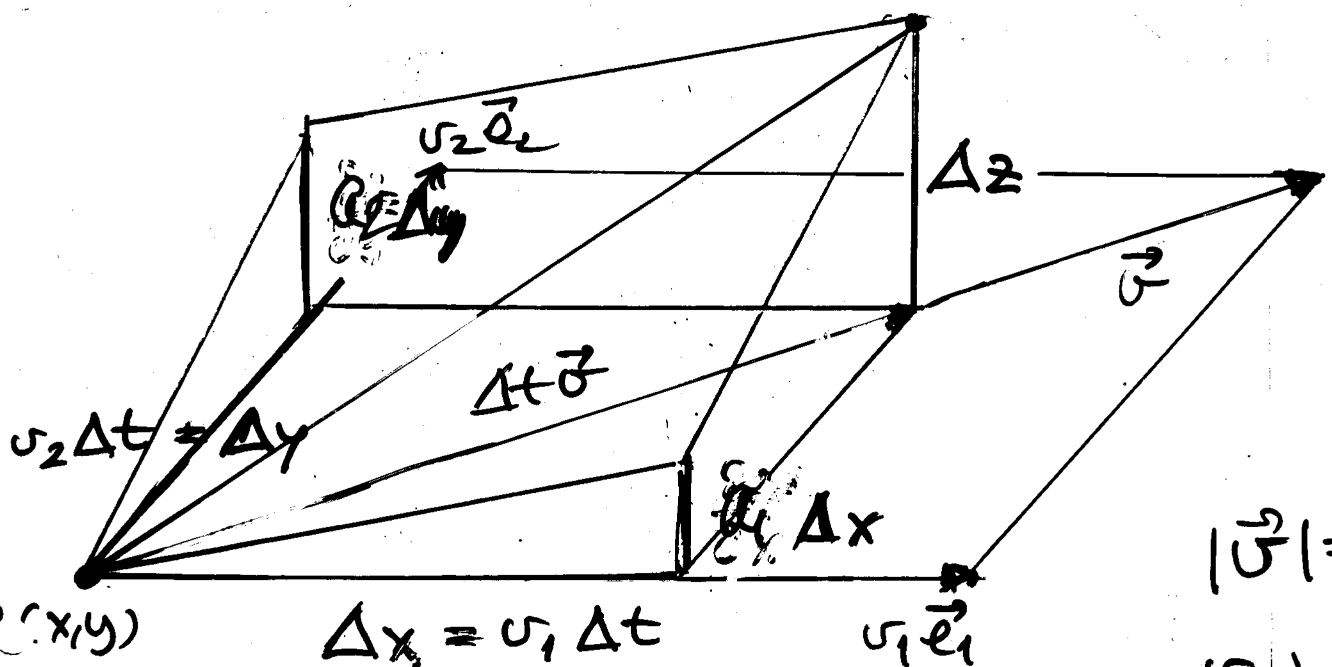
$$\Delta z = \underbrace{y\,\Delta x + x\,\Delta y}_{df\binom{x}{y}\binom{\Delta x}{\Delta y}} + \underbrace{\Delta x\,\Delta y}_{R\binom{\Delta x}{\Delta y}}$$

19.2.2





19.2.3 Richtungsableitung u. Gradient



$|\vec{v}| = 1$

$\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$

$$\lim_{\Delta t \to 0} \frac{\Delta z}{\Delta t} =$$

$$= \lim_{\Delta t \to 0} \quad a_1 \frac{\Delta x}{\Delta t} + a_2 \frac{\Delta y}{\Delta t} + \frac{R(\Delta x, \Delta y)}{\Delta t}$$

$$= a_1 v_1 + a_2 v_2$$

$$= \langle \operatorname{grad} f \,|\, \vec{v} \rangle$$ Ableitung in Richtung $\vec{v}$

$$\operatorname{grad} f = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$ Gradient

$\perp$ Höhenlinie

max in Richtung grad f

19.2.5

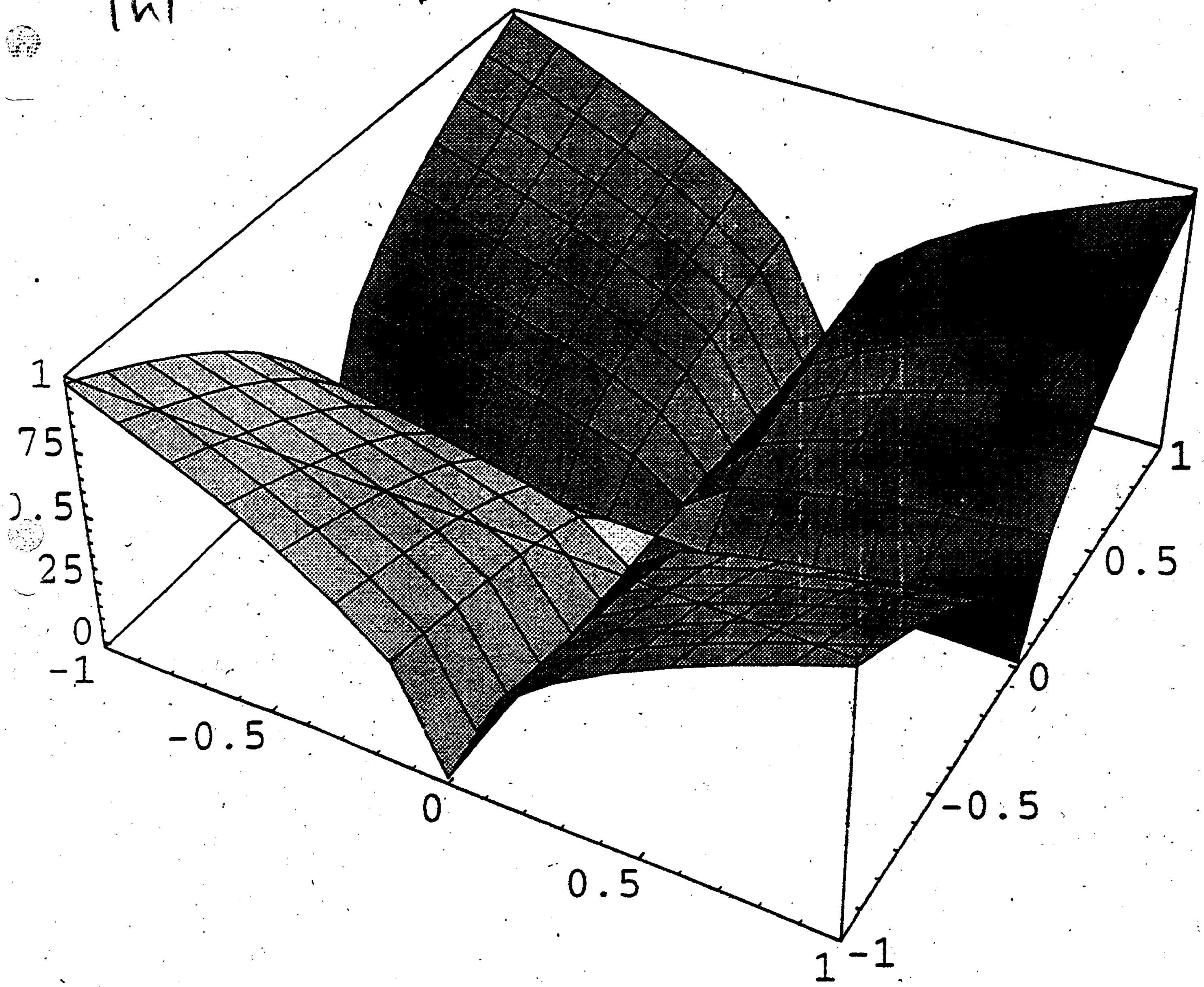
$z = f(x,y) = \sqrt{|xy|}$

$\frac{\partial z}{\partial x}(x,y) = \mp \frac{-y}{2\sqrt{|x|}}$ $x \neq 0$ $\frac{\partial z}{\partial y}(x,y) = \mp \frac{x}{2\sqrt{|y|}}$ $y \neq 0$

$\frac{\partial z}{\partial x}(x,0) = \frac{\partial z}{\partial y}(0,y) = 0$

nicht diffbar an $(0,0)$ - sonst $R(\vec{h}) = f(\vec{h})$

aber $\frac{1}{|\vec{h}|} R(\vec{h}) = \frac{t}{\sqrt{2}\,t} = \frac{1}{\sqrt{2}}$ für $\vec{h} = (t,t) \underset{t\to 0}{\to} \vec{0}$



19.2.5

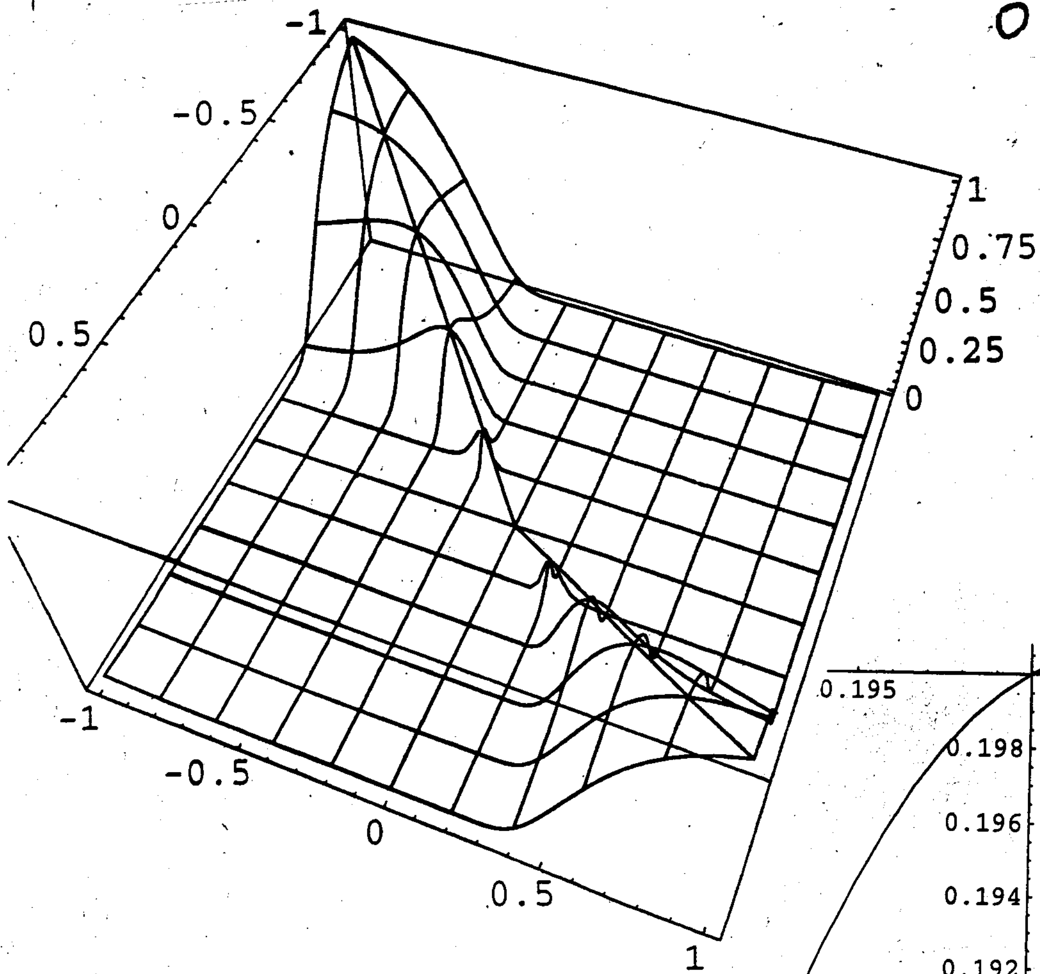
$$z = f(x,y) = \begin{cases} \sqrt{|xy|}\, e^{-\left(\frac{1}{x}-\frac{1}{5}\right)^2} & xy \neq 0 \\ 0 \end{cases}$$

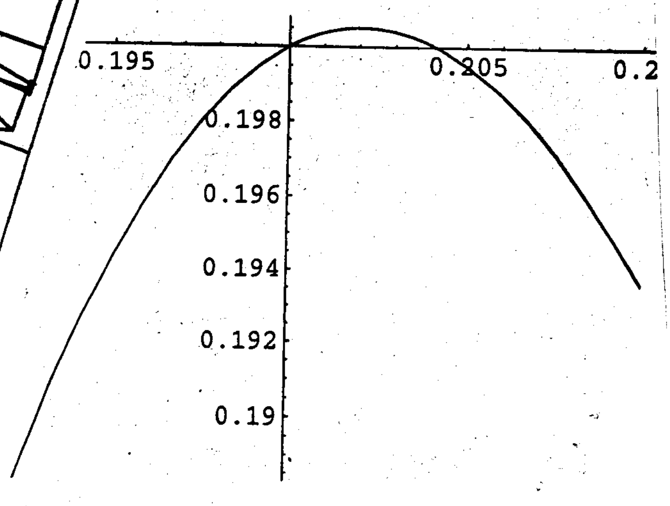
$\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ ex überall
stetig für $(x,y) \neq (0,0)$

keine Tangentialebene an $(0,0)$

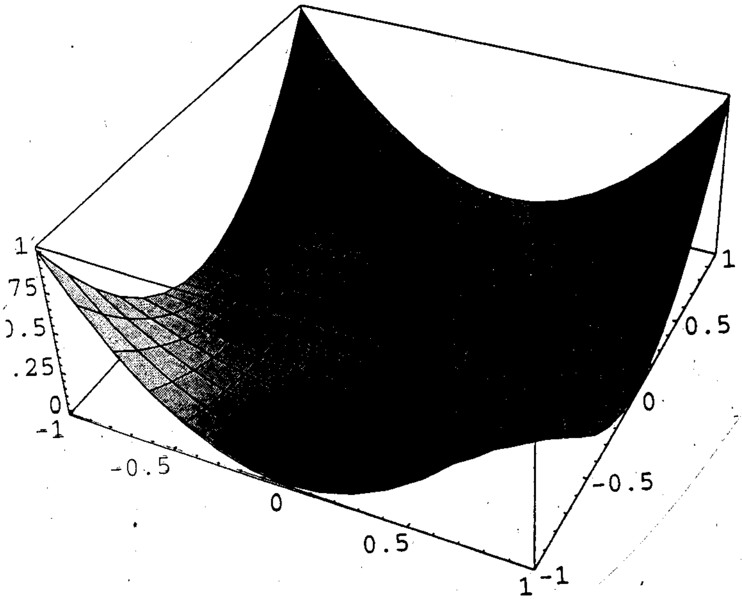
Beachte für $\gamma(t) = (t,t)$ $t \geq 0$

$f(\gamma(t)) = t$ $\quad (f\circ\gamma)'(0) = 1 \neq \sum \underbrace{\frac{\partial f}{\partial x_i}}_{0} \cdot \frac{\partial \gamma_i}{\partial t}$

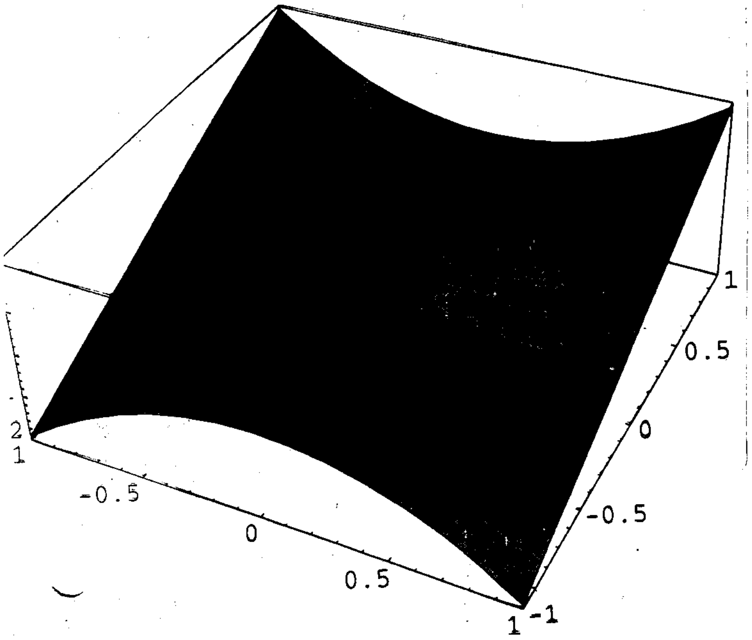


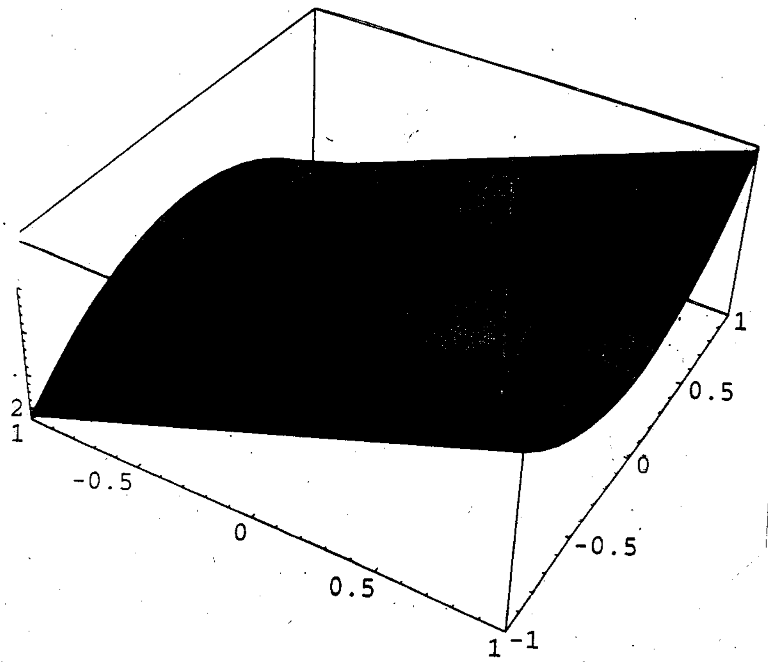


19.2.5/6



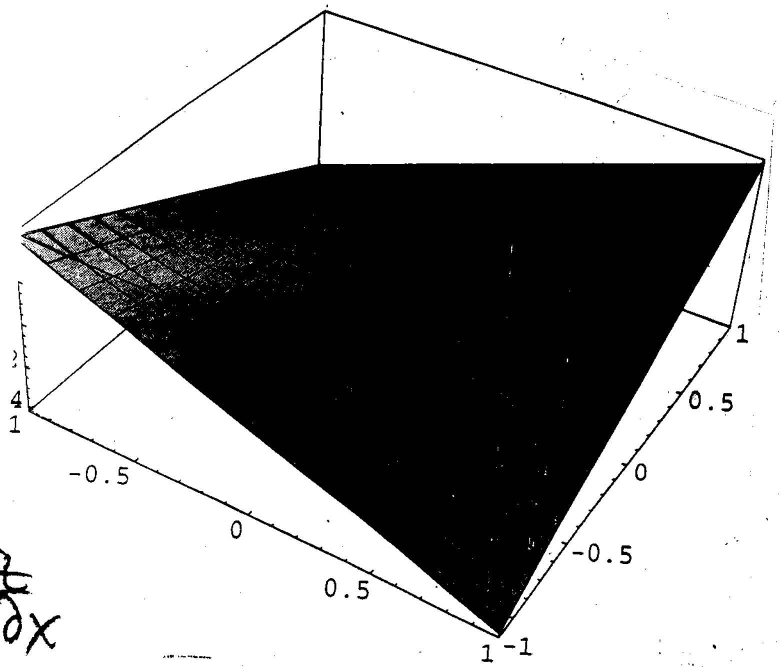
$z = x^2 + y^2$





$\frac{\partial z}{\partial y}$

$\frac{\partial z}{\partial x}$



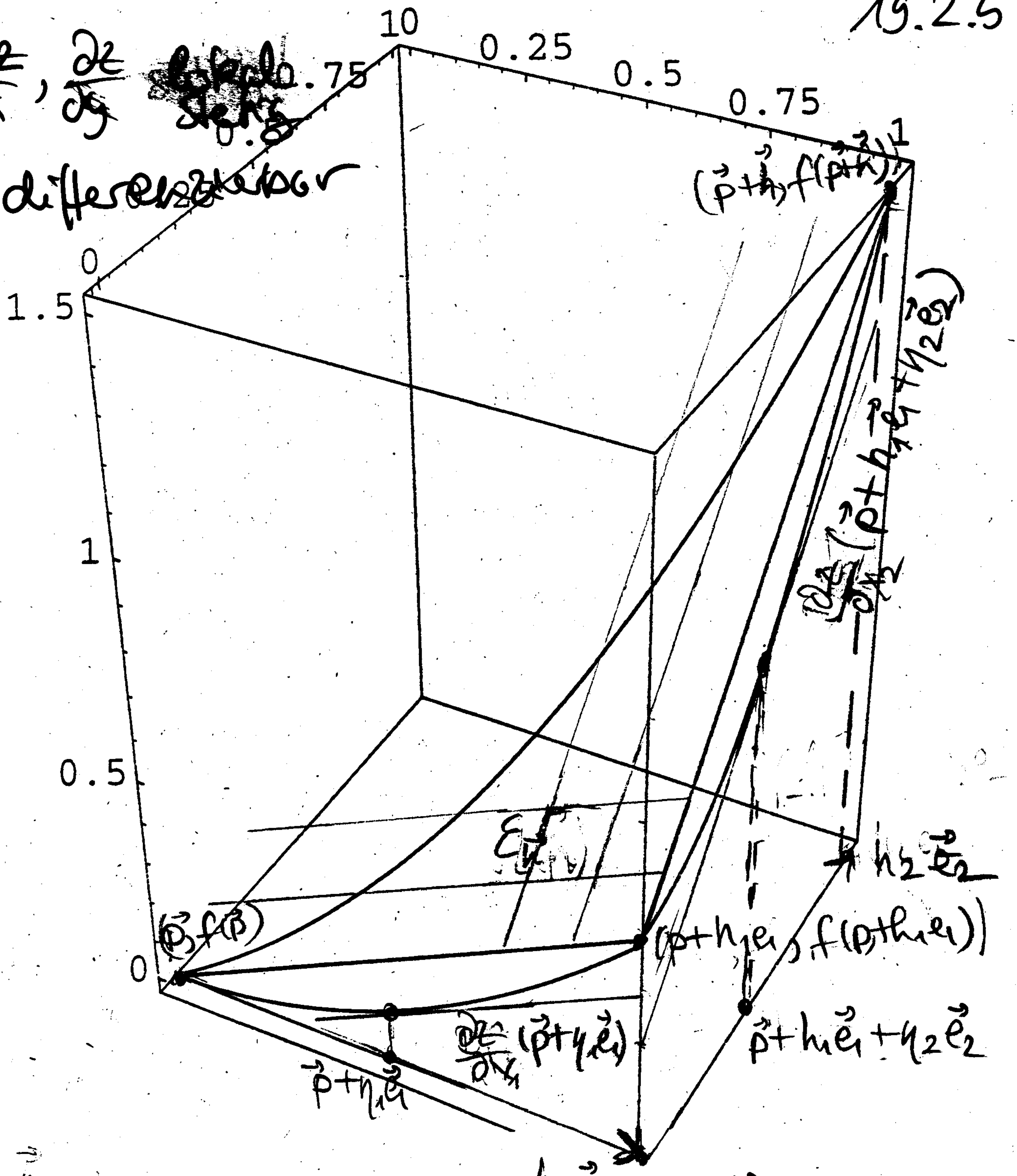
$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$$

$\vec p = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \vec h = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \Delta x_1 \\ \Delta x_2 \end{pmatrix}$ $\quad z = f\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{1}{2} x_1^2 + x_2^2$

19.2.5

$\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ stetig

$\Rightarrow$ differenzierbar



$$R\begin{pmatrix} \Delta x_1 \\ \Delta x_2 \end{pmatrix} = \overbrace{\frac{\partial z}{\partial x_1}(\vec p + \eta_1 \vec e_1)}^{c_1(\vec h)} \Delta x_1 + \overbrace{\frac{\partial z}{\partial x_2}(\vec p + \Delta x_1 \vec e_1 + \eta_2 \vec e_2)}^{c_2(\vec h)} \Delta x_2$$

$$- \underbrace{\frac{\partial z}{\partial x_1}(\vec p)}_{a_1} \Delta x_1 \quad - \underbrace{\frac{\partial z}{\partial x_2}(\vec p)}_{a_2} \Delta x_2 = \left\langle \begin{pmatrix} a_1 - c_1(\vec h) \\ a_2 - c_2(\vec h) \end{pmatrix} \middle| \vec h \right\rangle$$

$$R\begin{pmatrix} \Delta x_1 \\ \Delta x_2 \end{pmatrix} \Big/ \underbrace{\sqrt{\Delta x_1^2 + \Delta x_2^2}}_{\|\vec h\|} \longrightarrow 0 \text{ für } \begin{pmatrix} \Delta x_1 \\ \Delta x_2 \end{pmatrix} \to \vec 0$$

1926

oBdA $n=2,\ p=0\quad f(x,0)=f(0,y)=0$

$\frac{\partial z}{\partial x},\ \frac{\partial^2 z}{\partial x \partial y}$ ex. lokal stetig, $\frac{\partial^2 z}{\partial x \partial y} \to 0 \quad (x,y)\to 0$

$$\frac{1}{t}\,\frac{f(s,t)}{s} = \frac{1}{s}\,\frac{f(s,t)}{t} = \frac{1}{s}\,\frac{\partial z}{\partial y}(s,\tau)$$

$$= \frac{\partial^2 z}{\partial x \partial y}(\sigma,\tau) \to 0 \quad \text{für } (s,t)\to 0$$

also $(\sigma,\tau)\to 0$