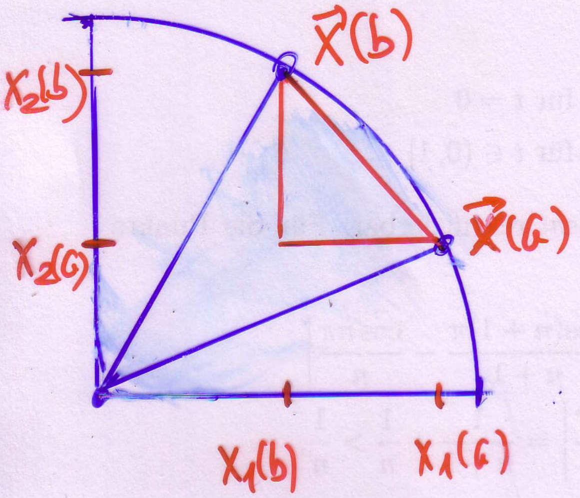
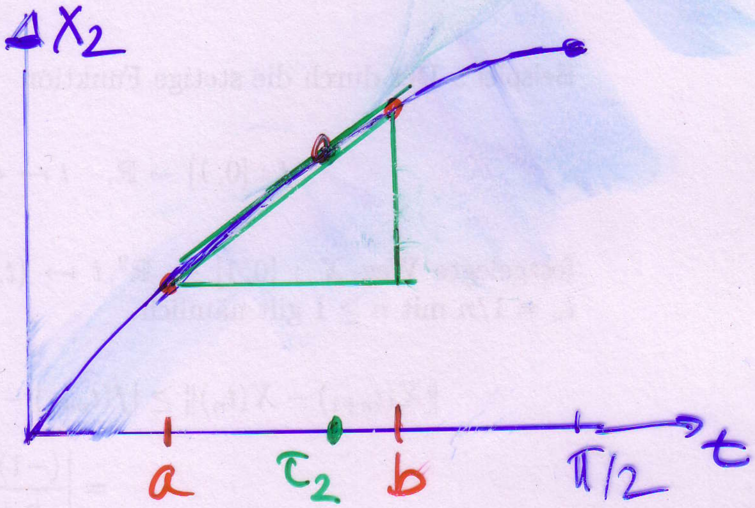
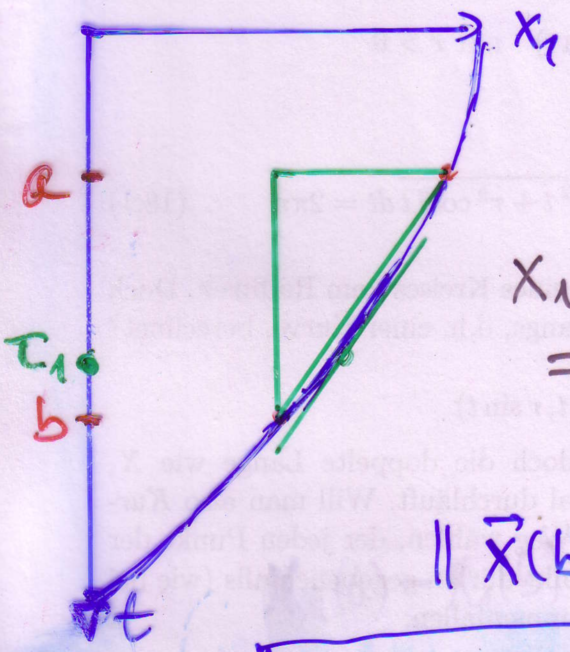
18.1.7. Mittelwertsatz





$$x_2(a) - x_2(b) = \frac{\partial x_2}{\partial t}(\tau_2)(b-a)$$



$$x_1(a) - x_1(b) = \frac{\partial x_1}{\partial t}(\tau_1)(b-a)$$

$$\|\vec{x}(b) - \vec{x}(a)\|$$

$$= \sqrt{[x_1(b) - x_1(a)]^2 + [x_2(b) - x_2(a)]^2}$$

$$= \sqrt{\left[\frac{\partial x_1}{\partial t}(\tau_1)(b-a)\right]^2 + \left[\frac{\partial x_2}{\partial t}(\tau_2)(b-a)\right]^2}$$

$$= \sqrt{\frac{\partial x_1}{\partial t}(\tau_1)^2 + \frac{\partial x_2}{\partial t}(\tau_2)^2}\,(b-a)$$

$\left\|\frac{\partial \vec{x}}{\partial t}\right\| = 1 \Rightarrow \searrow 1$ für $b - a \to 0$

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Lemma 18.5 $\vec{x}:[a,b]\to\mathbb{R}^n$ stetig diffbar

$$\left\|\frac{\partial \vec{x}}{\partial t}\right\| = 1 \Rightarrow$$

$$b-a = \sup \sum_{k=1}^{m} \|\vec{x}(t_k)-\vec{x}(t_{k-1})\| =: L$$

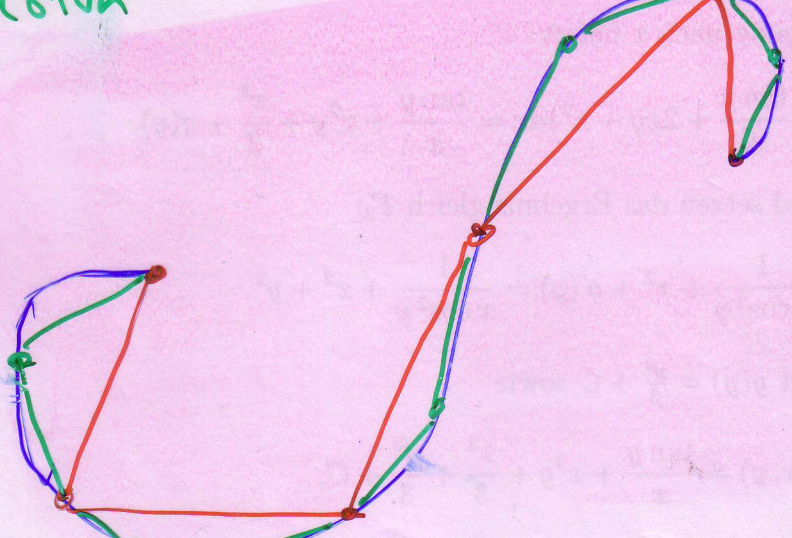
$$a=t_0<t_1<\dots<t_m=b$$

Bew.

$$L = \sum_{k=1}^{m} \underbrace{\sqrt{\sum_{i=1}^{n} \left[\frac{\partial x_i}{\partial t}(\tau_{ik})\right]^2}}_{\to 1}(t_k-t_{k-1})$$

$$L \to \sum_{k=1}^{m}(t_k-t_{k-1}) = b-a \quad \square$$

monoton



Korollar 18.7 $\vec{x}(t)$ $t\in[a,b]$ stetig diffbar

$$\left\|\frac{\partial \vec{x}}{\partial t}\right\| > 0 \rightarrow L(\vec{x}_{[a,b]}) = \int_a^b \left\|\frac{\partial \vec{x}}{\partial t}\right\| dt$$

Beweis $s = s(t) = \int_a^t \left\|\frac{\partial \vec{x}}{\partial t}(\tau)\right\| d\tau$

$$\frac{\partial s}{\partial t} = \left\|\frac{\partial \vec{x}}{\partial t}(t)\right\| > 0$$

$t = t(s)$ Umkehrfunktion

$$\frac{\partial \vec{x}}{\partial s} = \frac{\partial \vec{x}}{\partial t}\frac{\partial t}{\partial s} = \frac{\partial \vec{x}}{\partial t}\frac{1}{\frac{\partial s}{\partial t}} = \frac{\partial x}{\partial t}\frac{1}{\left\|\frac{\partial \vec{x}}{\partial t}\right\|}$$

$$\Rightarrow \left\|\frac{\partial \vec{x}}{\partial s}\right\| = 1$$

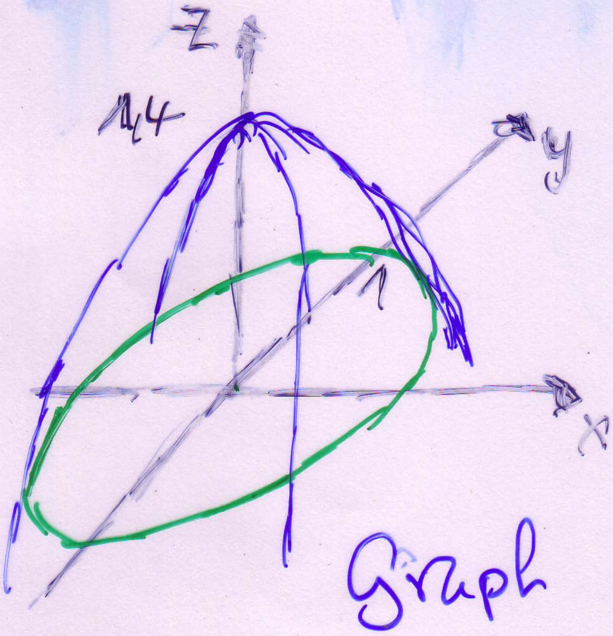
$$L(\vec{x}_{[a,b]}) = L(\vec{x}(t(s)))_{[s(a),s(b)]}$$

$$= s(b) - \underbrace{s(a)}_{=0} = \int_a^b \left\|\frac{\partial \vec{x}}{\partial t}\right\| dt$$

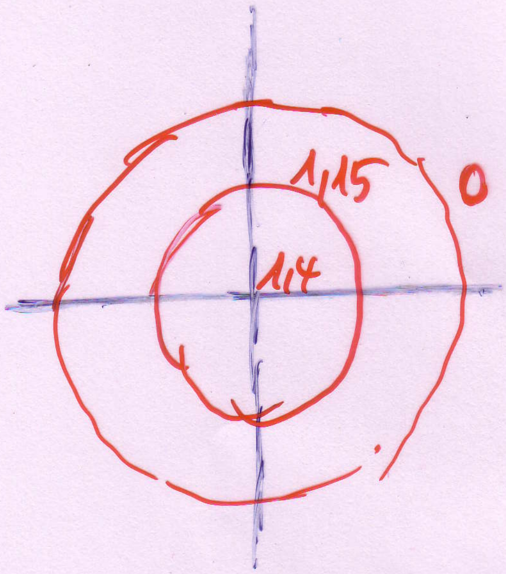
18.2.1 Skalarenfeld

$$f : D \to \mathbb{R} \qquad D \subseteq \mathbb{R}^n$$

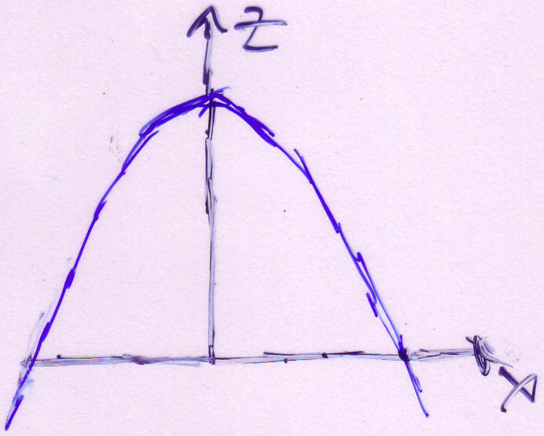
$$z = f\begin{pmatrix} x \\ y \end{pmatrix} = 1{,}4 - (x^2 + y^2) \qquad n = 2$$



Graph

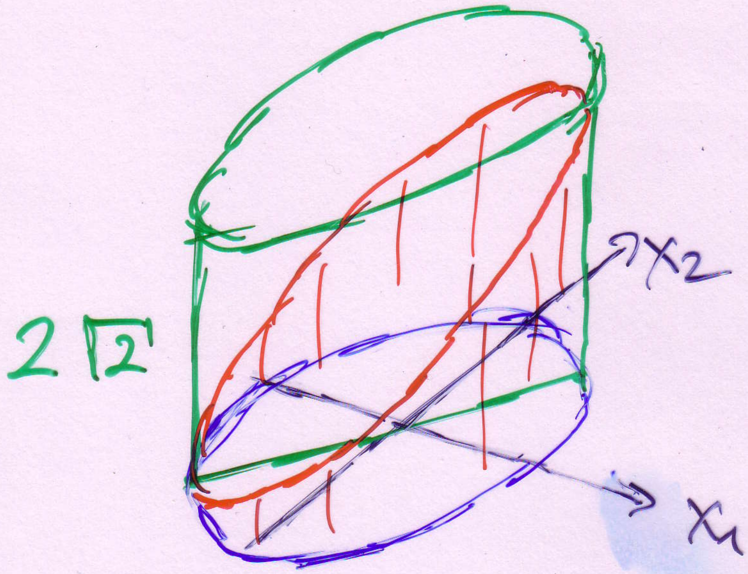


Höhenlinien



Schnitt mit Koordinaten ebene

18.22 Wegintegral im Skalarenfeld



$$\vec{x}(t) = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} \quad t \in [0, 2\pi]$$

$$f(\vec{x}) = x_1 + x_2 + \sqrt{2} \qquad \left\| \frac{\partial \vec{x}}{\partial t} \right\| = 1$$

$$\int_0^{2\pi} f(\vec{x}(t))\, dt$$

$$= \int_0^{2\pi} \cos t + \sin t + \sqrt{2} \;\, dt$$

$$= 2\sqrt{2}\,\pi$$

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$$\vec{x}(t) = \begin{pmatrix} \cos 1/2t \\ \sin 1/2t \end{pmatrix} \qquad t \in [0, 2\pi]$$

$$f(\vec{x}) = x_1 + x_2 + \sqrt{2}$$

$$\frac{\partial \vec{x}}{\partial t} = \frac{1}{2} \begin{pmatrix} -\sin 1/2t \\ \cos 1/2t \end{pmatrix}$$

$$\left\| \frac{\partial \vec{x}}{\partial t} \right\| = \frac{1}{2}$$

$$\int_a^b f(\vec{x}(t)) \left\| \frac{\partial \vec{x}}{\partial t} \right\| dt$$

$$= \int_0^{2\pi} \left(\cos \frac{1}{2}t + \sin \frac{1}{2}t + \sqrt{2} \right) \frac{1}{2} \, dt$$

$$= \left[2\sin \frac{1}{2}t - 2\cos \frac{1}{2}t + \sqrt{2}t \right) \frac{1}{2} \Big]_0^{2\pi}$$

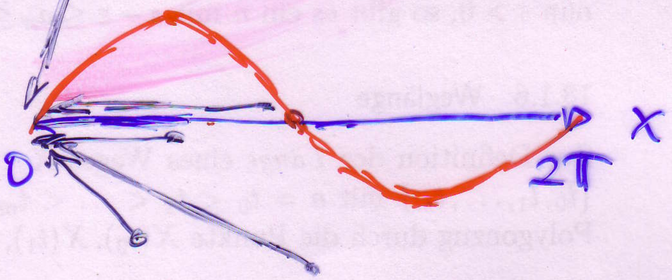
18.2.3 Wegintegral im Vektorfeld

$F(\vec{x}) \in \mathbb{R}^n, \quad \vec{x}(t) \in \mathbb{R}^n \quad t \in [a,b]$

$$\int_a^b \left\langle F(\vec{x}(t)) \,\middle|\, \frac{\partial \vec{x}}{\partial t}(t) \right\rangle dt$$

$$F\begin{pmatrix} x \\ y \end{pmatrix} = -\begin{pmatrix} x \\ y \end{pmatrix}$$

$\vec{x}(t) = \begin{pmatrix} t \\ 0 \end{pmatrix} \quad t \in [0, 2\pi]$



$\vec{x}(t) = \begin{pmatrix} t \\ \sin t \end{pmatrix} \quad t \in [0, 2\pi]$

$$\frac{\partial \vec{x}}{\partial t} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \int_0^{2\pi} \left\langle -\begin{pmatrix} t \\ 0 \end{pmatrix} \middle| \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle dt = \int_0^{2\pi} -t\,dt = 2\pi^2$$

$$\frac{\partial x}{\partial t} = \begin{pmatrix} 1 \\ \cos t \end{pmatrix} \qquad \int_0^{2\pi} \left\langle -\begin{pmatrix} t \\ \sin t \end{pmatrix} \middle| \begin{pmatrix} 1 \\ \cos t \end{pmatrix} \right\rangle dt$$

$$= \int_0^{2\pi} -t - \sin t \cos t \; dt = \left[-\frac{1}{2}t^2 - \frac{1}{2}\sin^2 t \right]_0^{2\pi}$$

$$= -2\pi^2$$

$\vec{x}(t) = \begin{pmatrix} 2t \\ 0 \end{pmatrix} \qquad \frac{\partial \vec{x}}{\partial t} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$

$$\int_0^{2\pi} \left\langle -\begin{pmatrix} t \\ 0 \end{pmatrix} \middle| \begin{pmatrix} 2 \\ 0 \end{pmatrix} \right\rangle dt = \int_0^{2\pi} -2t\,dt = -4\pi^2$$

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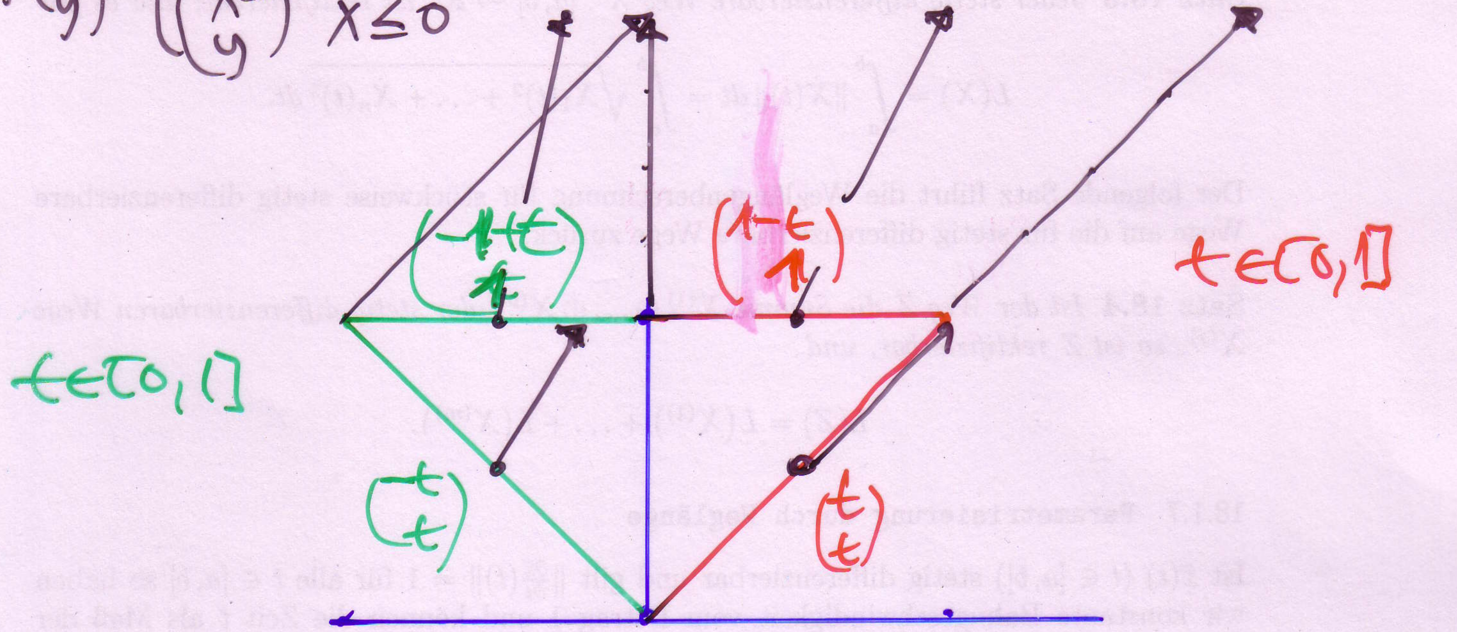
$$F\binom{x}{y} = \begin{cases} \binom{x}{y} & x \geq 0 \\ \binom{x^2}{y} & x \leq 0 \end{cases}$$

18.2.3 Wegintegral im Vektorfeld



$$\int_0^1 \langle \binom{t}{t} | \binom{1}{1} \rangle dt + \int_0^1 \langle \binom{1-t}{1} | \binom{-1}{0} \rangle dt$$

$2t \qquad t-1$

$$= \int_0^1 3t - 1 \, dt = \frac{1}{2}$$

$$\int_0^1 \langle \binom{(-t)^2}{t} | \binom{-1}{1} \rangle dt + \int_0^1 \langle \binom{(-1+t)^2}{1} | \binom{1}{0} \rangle dt$$

$$= \int_0^1 -t^2 + t + (t-1)^2 \, dt$$

$$= -\frac{1}{3} + \frac{1}{2} - \frac{1}{3} = -\frac{1}{6}$$