Variationsrechnung

7. Übung, Lösungsvorschlag

Gruppenübung

G 1 Zeigen Sie, wenn

$$a(x,y)u_x(x,y) + b(x,y)u_y(x,y) = c(x,y)$$

mit

$$\begin{aligned} x = x(s), &\quad y = y(s), \\ u(s) = u(x(s), y(s)), &\quad c(s) = c(x(s), y(s)) \\ x'(s) = a(x(s), y(s)), &\quad y'(s) = b(x(s), y(s)) \end{aligned}$$

und

$$x(0) = x_0, \quad y(0) = y_0$$

gilt, dann

$$u'(s) = c(s)$$

mit

$$u(0) = u(x_0, y_0)$$

gilt.

$$\frac{d}{dt}u(s) = \frac{d}{dt}u(x(s), y(s)) = u_x x' + u_y y' = a(x,y)u_x + b(x,y)u_y = c(x(s), y(s)) = c(s).$$

G 2 **[Clairautsche Differentialgleichung]**

$$xu'(x) - u(x) = f(u'(x)) \tag{$\star$}$$

1. Transformieren Sie ($\star$) mit Hilfe der Legendretransformation in

$$u^\star(y) = f(y).$$

Hinweis: $f^\star(\nabla f(x)) = x\nabla f(x) - f(x)$.

From the formula in the hint we have that

$$u^\star(u'(x)) = xu'(x) - u(x) = f(u'(x))$$

and so taking $y = u'(x)$ *proves the desired equality.*

2. Lösen Sie ($\star$) mit

   (a) $f(x) = x^2$,
   Since the function f *is convex, we have that* $(f^\star)^\star = f$. *Therefore from the equation*

$$u^\star(y) = f(y)$$

we can compute that

$$u(x) = f^\star(x).$$

But

$$f^\star(x) = \sup_{y\in\mathbb{R}}(xy - f(y)) = \frac{1}{4}x^2 = u(x).$$

(b) $f(x) = x^4$,

$$u(x) = 3\left(\frac{x}{4}\right)^{\frac{4}{3}}$$

(c) $f(x) = e^x$,

$$u(x) = x(\ln x - 1), \quad x > 0$$

(d) $f(x) = \ln x$, $x > 0$.
This should be $f(x) = -\ln x$ with $x > 0$, since this function is convex. Then we have

$$u(x) = -1 - \ln(-x), \quad x < 0$$

Hausübung

H 1 Lösen Sie die 2-dimensionelle Clairautsche Gleichung

$$xu_x + yu_y - u = f(u_x, u_y)$$

mit $f(x) = |x|^2$.

The procedure is exactly the same as above, so that $u^\star(y) = f(y)$. Since the function is convex, $u(x) = f^\star(x)$. Therefore

$$u(x) = f^\star(x) = \frac{1}{4}|x|^2$$

or, equivalently

$$u(x, y) = \frac{1}{4}\left(x^2 + y^2\right).$$