Variationsrechnung

6. Übung, Lösungsvorschlag

Gruppenübung

G 1 [Berührungstransformation]

1. Es sei eine stetig differenzierbare Vektorfeld

$$(x, p, z) \mapsto (X(x, p, z), P(x, p, z), Z(x, p, z)) \colon \mathbb{R}^3 \to \mathbb{R}^3$$

und eine stetige Funktion

$$(x, y, z) \mapsto \rho(x, y, z) \colon \mathbb{R}^3 \to \mathbb{R}$$

gegeben. Man bestimme Bedingungen an X, P, Z und ρ, so dass für alle stetig differenzierbaren Funktionen

$$t \mapsto (x(t), p(t), z(t)) \colon \mathbb{R} \to \mathbb{R}^3$$

die Gleichung

$$\frac{dZ}{dt} - P\frac{dX}{dt} = \rho\left(\frac{dz}{dt} - p\frac{dx}{dt}\right) \qquad (\star)$$

erfüllt ist, wobei

$$\begin{aligned} X(t) =& X(x(t), p(t), z(t)), \\ P(t) =& P(x(t), p(t), z(t)), \\ Z(t) =& Z(x(t), p(t), z(t)), \\ \rho =& \rho(x(t), p(t), z(t)) \end{aligned}$$

seien.

We will write $Z_x = \frac{\partial Z}{\partial x}$ and $\dot{x} = \frac{dx}{dt}$ and so on:

$$\begin{aligned} \frac{dZ}{dt} - P\frac{dX}{dt} =& Z_x\dot{x} + Z_p\dot{p} + Z_z\dot{z} - P(X_x\dot{x} + X_p\dot{p} + X_z\dot{z}) \\ =& (Z_x - PX_x)\dot{x} + (Z_p - PX_p)\dot{p} + (Z_z - PX_z)\dot{z} = \rho(\dot{z} - p\dot{x}), \end{aligned}$$

the last equality follows from $(\star)$. Since x, p, z are arbitrary functions, we can take $x(t) = x_0 + t$, $p(t) = p_0$ and $z(t) = z_0$ and obtain that

$$Z_x(x_0, p_0, z_0) - P(x_0, p_0, z_0)X_x(x_0, p_0, z_0) = \rho(x_0, p_0, z_0) \iff Z_x - PX_x = \rho$$

for all $(x_0, p_0, z_0) \in \mathbb{R}^3$. Similarly

$$Z_p - PX_p = 0$$

and

$$Z_z - PX_z = -\rho p.$$

These three equations are the required conditions.

2. Man gebe ein explizites Beispiel für die Abbildungen

$$(x, p, z) \mapsto (X, P, Z) \quad \text{und} \quad (x, p, z) \mapsto \rho$$

an, so dass $(\star)$ gilt. (Man denke an die Legendretransformation). Gibt es weitere Beispiele?

The simplest transformation is

$$Z = z,\ P = p,\ X = x,\ \rho = 1$$

and it is easy to check that $(\star)$ *holds. A similar example is*

$$Z = cz,\ P = \frac{c}{a}p,\ X = ax,\ \rho = c,$$

where c *and* $a \neq 0$ *are some constants.*
The Legendre transform provides us with

$$Z = xp - z,\ P = x,\ X = p.$$

3. Sei $f \in C^1(\mathbb{R}, \mathbb{R})$ gegeben, sei

$$z(t) = f(x(t)), \quad p(t) = f'(x(t)).$$

Zeige, dass dann

$$\frac{dZ(t)}{dt} - P(t)\frac{dX(t)}{dt} = 0$$

gilt.

This is an immediate consequence of $(\star)$ *and the computation of* $\dot{z} - p\dot{x} = 0$*.*