Variationsrechnung

5. Übung, Lösungsvorschlag

Hausübung

H 1 Sei $f(x) = |x|^2$ für $x \in \mathbb{R}^n$.

1. Berechnen Sie die Legendretransformierte $f^\star$ von f.

   We have $f^\star(y) = \sup_{x\in\mathbb{R}^n}(x \cdot y - f(x))$ *and so*

   $$f^\star(y) = \sup_{x\in\mathbb{R}^n}(x \cdot y - |x|^2).$$

   Observe that the above supremum is attained if x *is parallel to* y *(then* $x \cdot y$ *has a maximum, while* $|x|$ *doesn't depend on the angle between* x *and* y*). Therefore* $x = \lambda y$*, where* $\lambda > 0$ *is the maximum of the function*

   $$g(\lambda) = \lambda y \cdot y - |\lambda y|^2 = (\lambda - \lambda^2)|y|^2.$$

   This maximum is attained for $\lambda = \frac{1}{2}$ *and so* $x = \frac{1}{2}y$ *and* $f^\star(y) = \frac{1}{2}|y|^2 - \frac{1}{4}|y|^2 = \frac{1}{4}|y|^2$.

2. Zeigen Sie dass $f^\star(\nabla f(x)) = \nabla f(x) \cdot x - f(x)$ gilt.

   Observe that $\nabla f(x) = 2x$ *and so*

   $$f^\star(\nabla f(x)) - \nabla f(x) \cdot x + f(x) = \frac{1}{4}|2x|^2 - 2x \cdot x + |x|^2 = 0.$$

H 2 Sei $\overline{K} = \{|x| \le 1\} \subset \mathbb{R}^n$ eine Kugel und sei $f\colon \mathbb{R}^n \to [0, +\infty]$ definiert durch

$$f(x) = \begin{cases} 0, & x \in \overline{K} \\ +\infty, & x \in \mathbb{R}^n \setminus \overline{K}. \end{cases}$$

1. Zeigen Sie, dass f konvex ist.

   This follows from convexity of K *and the fact that* $f = \infty$ *on* $\mathbb{R}^n \setminus \overline{K}$.

2. Berechnen Sie das Subdifferential ∂f von f.

   The definition of subdifferential is: $v \in \partial f(x)$*, when*

   $$\forall y \in \mathbb{R}^n \;\; f(y) \ge f(x) + (v, y - x).$$

   Observe that if $x \in \mathbb{R}^n \setminus \overline{K}$*, then* $f(x) = \infty$ *and taking* $y \in K$ *the above inequality cannot hold (since* $f(y) = 0$*). Therefore,* $\partial f(x) = \emptyset$ *for* $x \in \mathbb{R}^n \setminus \overline{K}$*. Observe also, that for* $x \in \operatorname{Int} K$ *we can take* $\rho > 0$ *so small that* $B(x, \rho) \subset \operatorname{Int} K$*. Then, taking* $y \in B(x, \rho)$ *we have that* $w = y - x \in B(0, \rho)$ *and so*

   $$0 \ge 0 + (v, w)$$

   which implies that $v = 0$*. Therefore* $\partial f(x) = \{0\}$ *for* $x \in \operatorname{Int} K$*.*
   It remains to compute what is $\partial f(x)$ *for* $x \in \partial K$*. Observe that it is only sufficient to take* $y \in K$*. Then we have*

   $$0 \ge (v, y - x) \iff (v, x - y) \ge 0$$

   for all $y \in K$*. This means that the angle between* v *and* $x - y$ *is between* $-\frac{\pi}{2}$ *and* $\frac{\pi}{2}$ *and so* $\partial f(x)$ *is the half-space, tangent to* K *at x, not including* K*.*