Variationsrechnung

4. Übung, Lösungsvorschlag

Gruppenübung

G 1 Let $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$ for $1 \le p \le \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Define the *convolution* of f and g by the formula

$$(f \star g)(x) = \int_{\mathbb{R}^n} f(x-y)g(y)dy.$$

1. Prove that $f \star g = g \star f$.

   This follows by a change of variables $z = x - y$.

2. For $f = \chi_{[0,1]}$ compute $f \star f$ and sketch its graph.

   We have

   $$g(x) = (f \star f)(x) = \int_{\mathbb{R}} f(x-y)f(y)dy = \int_0^1 \chi_{[0,1]}(x-y)dy.$$

   We see now that $g(x) = 0$ if $x < 0$ or $x > 2$. For $x \in [0,1]$ we have

   $$g(x) = \int_0^x dy = x$$

   and for $x \in [1,2]$

   $$g(x) = \int_{x-1}^1 dy = 2 - x.$$

   In particular, observe that g is continuous.

G 2 Let $1 \le p < \infty$.

1. Prove that the space $C_0(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$.
   Hint: Start by approximating characteristic functions by continuous functions.

   Every function $f \in L^p(\mathbb{R}^n)$ with $1 \le p < \infty$ is an L^p-limit of simple functions f_n such that $|f_n| \nearrow |f|$ pointwise a.e. From this it follows that it is sufficient to approximate characteristic functions in the L^p norm. Let $A \subset \mathbb{R}^n$ be an open bounded set. Take $B \subset \mathbb{R}^n$ open and bounded such that $A \subset \overline{A} \subset B$ and $|B \backslash A| < \epsilon$. Then, from Uryhson's lemma, there exists a continuous function g such that $g \equiv 1$ on A, $g \equiv 0$ on $\mathbb{R}^n \setminus B$ and $0 \le g(x) \le 1$ for all $x \in \mathbb{R}^n$. This function approximates the characteristic function χ_A in L^p because

   $$\|\chi_A - g\|^p_{L^p(\mathbb{R}^n)} = \int_{\mathbb{R}^n} |\chi_A(x) - g(x)|^p dx = \int_{B\backslash A} |g(x)|^p dx \le \int_{B\backslash A} dx = |B \setminus A| < \epsilon.$$

   Now, for a simple function $f = \sum_{k=0}^N a_k \chi_{A_k}$ it is sufficient to approximate each χ_{A_k} with g as above and sets B_k such that $A_k \subset \overline{A}_k \subset B_k$ and $\sum_{k=0}^N |B_k \setminus A_k| < \epsilon$.

2. Prove that for $f \in L^p(\mathbb{R}^n)$

   $$\int_{\mathbb{R}^n} |f(x+y) - f(x)|^p dx \to 0$$

   when $|y| \to 0$.

Since $C_0(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$, it is sufficient to prove the statement for $g \in C_0(\mathbb{R}^n)$. Obviously, we have the pointwise convergence $g(x+y) \to g(x)$ for $|y| \to 0$, or, equivalently $|g(x+y) - g(x)|^p \to 0$ for $|y| \to 0$. Moreover, since g has compact support K, it is bounded on K and in fact in all of $\mathbb{R}^n$. Also

$$\|g(\cdot + y) - g(\cdot)\|_{L^p(\mathbb{R}^n)} \leq 2\|g\|_{L^p(\mathbb{R}^n)} \leq \sup_{x \in K} |g(x)| \cdot |K|.$$

Applying Lebesgue's dominated convergence theorem the proof is finished.

Hausübung

H 1 Let $\rho \in C_0^\infty(\mathbb{R}^n)$ be such that

$$\rho(x) \geq 0, \qquad \int_{\mathbb{R}^n} \rho(x) dx = 1$$

and take $\rho_\epsilon(x) = C_\epsilon \rho\left(\frac{x}{\epsilon}\right)$ for $\epsilon > 0$.

1. Compute C_ϵ for which $\int_{\mathbb{R}^n} \rho_\epsilon(x) dx = 1$.

   By a change of variables $y = \frac{x}{\epsilon}$ we get that $C_\epsilon = \epsilon^{-n}$.

2. Take $1 \leq p < \infty$. Prove that

   $$f \star \rho_\epsilon \to f \quad \text{in } L^p(\mathbb{R}^n)$$

   when $\epsilon \to 0$ and from this conclude that the space $C_0^\infty(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$.

   Since the functions $f \star \rho_\epsilon$ are of class $C_0^\infty(\mathbb{R}^n)$, the proof of convergence will immediately yield the fact that C_0^∞ is dense in L^p for $1 \leq p < \infty$.
   We will prove convergence for $f \in C_0(\mathbb{R}^n)$ and then the result can be extended, since C_0 is dense in L^p as has been proved above. We use the fact that $\int_{\mathbb{R}^n} \rho_\epsilon(x) dx = 1$ and that $\rho_\epsilon \geq 0$:

   $$\begin{aligned} |(f \star \rho_\epsilon)(x) - f(x)| &= \left| \int_{\mathbb{R}^n} f(x-y) \rho_\epsilon(y) dy - f(x) \right| \\ &= \left| \int_{\mathbb{R}^n} [f(x-y) - f(x)] \rho_\epsilon(y) dy \right| \\ &\leq \int_{\mathbb{R}^n} |f(x-y) - f(x)| \rho_\epsilon(y) dy \\ &= \int_{\mathbb{R}^n} |f(x - \epsilon z) - f(x)| \rho(z) dz \to 0 \end{aligned}$$

   when $\epsilon \to 0$ (this follows from **G 2***). In a similar fasion, using Hölder's inequality, we prove that $\|f \star \rho_\epsilon - f\|_{L^p(\mathbb{R}^n)}$ is bounded. From this it follows the convergence $f \star \rho_\epsilon \to f$ in L^p.*

3. Show that the above convergence does not hold for $p = \infty$.

   Take $Q = [0,1]^n$ and $f = \chi_Q$. Then $f \in L^\infty$, but no sequence of C_0^∞ (or even C_0) functions can approximate f in the L^∞ norm, because then f would have to be continuous (since the uniform limit of continuous functions is continuous).