Variationsrechnung

3. Übung, Lösungsvorschlag

Gruppenübung

G 1

1. *We have $F(0,\xi) = (|\xi| - 1)^2$ and this is obviously not convex: $F(0,-1) = F(0,1) = 0$, but $F(0,0) = 1$ and so it's not true that*
$$F\left(\frac{1}{2}(0,-1) + \frac{1}{2}(0,1)\right) \leq \frac{1}{2}F(0,-1) + \frac{1}{2}F(0,1).$$

2. *Take*
$$u_1(x) = \begin{cases} x & \text{for } 0 \leq x < \frac{1}{2} \\ 1 - x & \text{for } \frac{1}{2} \leq x \leq 1. \end{cases}$$
Then u_1 is piecewise continuously differentiable on $[0,1]$ and
$$\mathcal{I}[u_1] = \int_0^1 u_1^2(x)dx = 2\int_0^{\frac{1}{2}} xdx = \frac{1}{4}.$$
Now take
$$u_2(x) = \begin{cases} x & \text{for } 0 \leq x < \frac{1}{4} \\ \frac{1}{2} - x & \text{for } \frac{1}{4} \leq x < \frac{1}{2} \\ x - \frac{1}{2} & \text{for } \frac{1}{2} \leq x < \frac{3}{4} \\ 1 - x & \text{for } \frac{3}{4} \leq x \leq 1. \end{cases}$$
Observe that $u_2(x) = \frac{1}{2}u_1(2x)$ if we extend u_1 periodically onto the set $[0,2]$. Then
$$\mathcal{I}[u_2] = \int_0^1 u_2^2(x)dx = \frac{1}{2}\int_0^1 u_1^2(2x)dx = \frac{1}{4}\int_0^2 u_1^2(x)dx = \frac{1}{2}\int_0^1 u_1^2(x)dx = \frac{1}{2}\mathcal{I}[u_1].$$
Similarly, if we extend u_1 onto the whole half-line $\mathbb{R}_+$ periodically, then taking $u_{n+1}(x) = 2^{-n}u_1(2^n x)$ we see that
$$\mathcal{I}[u_{n+1}] = 2^{-n}\mathcal{I}[u_1] \to 0$$
as $n \to \infty$. Obviously each u_n is piecewise continuously differentiable and also $u_n \to 0 = u$ pointwise on $[0,1]$ (even uniformly, because $\sup_{x\in[0,1]} |u_n(x)| = 2^{-n} \to 0$). But
$$\mathcal{I}[0] = \int_0^1 (0-1)^2 dx = 1$$
and so we have $\mathcal{I}[u] \neq \lim_{n\to\infty} \mathcal{I}[u_n]$.

3. *Fix $x < y \in [a,b]$. We need to prove that for all $\lambda \in (0,1)$ there holds*
$$f((1-\lambda)x + \lambda y) \leq (1-\lambda)f(x) + \lambda)f(y).$$
This is true for $\lambda = \frac{1}{2}$, from the assumption. Take $\lambda = \frac{1}{4}$. Then the point $(1-\lambda)x+\lambda y = \frac{3x+y}{4}$ is the midpoint between x and $\frac{x+y}{2}$ and so:
$$\begin{aligned} f\left(\frac{3}{4}x + \frac{1}{4}y\right) =& f\left(\frac{1}{2}x + \frac{1}{2}\frac{x+y}{2}\right) \leq \frac{1}{2}f(x) + \frac{1}{2}f\left(\frac{x+y}{2}\right) \\ \leq& \frac{1}{2}f(x) + \frac{1}{4}f(x) + \frac{1}{4}f(y) = \frac{3}{4}f(x) + \frac{1}{4}f(y). \end{aligned}$$
Similarly we prove for $\lambda = \frac{3}{4}$. By dividing these intervals into two pieces we prove the inequality for $\lambda = \frac{k}{2^n}$ for any $n = 1, 2, ...$ and $k = 0, ..., n$. But the set
$$L = \{(1-\lambda)x + \lambda y \colon \exists n \in \mathbb{N}\ \exists k \in \{0, ..., n\}\ \lambda = \frac{k}{2^n}\}$$
is dense in the interval $[x,y]$. From this density and continuity of f we obtain the desired inequality for any $\lambda \in (0,1)$.