Variationsrechnung

2. Übung, Lösungsvorschlag

Gruppenübung

G 1 Let
$$D = \{u \in C^1[0,1] \colon u(0) = a \wedge u(1) = b\}$$
and let $\mathcal{I}[u]$ denote the length of curve $x \mapsto (x, u(x))$ for $u \in D$.

1. Write $\mathcal{I}[u]$ in integral form.

$$\mathcal{I}[u] = \int_0^1 \sqrt{1 + (u'(x))^2}dx.$$

2. Let $\phi \in C^0_\infty(0,1)$ and let $f(t) = \mathcal{I}[u + t\phi]$ for $t \in \mathbb{R}$. Observe that $u + t\phi \in D$ and then compute $f'(t)$.

$$f'(t) = \frac{d}{dt} \int_0^1 \sqrt{1 + (u'(x) + t\phi'(x))^2}dx = \int_0^1 \frac{\phi'(x)(u'(x) + t\phi'(x))}{\sqrt{1 + (u'(x) + t\phi'(x))^2}}dx \qquad (\star)$$

3. Suppose $u \in D$ is such that
$$\mathcal{I}[u] = \min_{v \in D} \mathcal{I}[v].$$
What condition must f satisfy? Is this condition also sufficient? Write a proper differential equation for u and solve it.

The condition is $f'(0) = 0$ and it is usually not sufficient for the minimum to exist. Notice that
$$f'(0) = \int_0^1 \frac{\phi'(x)u'(x)}{\sqrt{1 + (u'(x))^2}}dx$$
and so, after we differentiate by parts and note that $\phi(0) = \phi(1) = 0$ (since ϕ has compact support):
$$f'(0) = -\int_0^1 \phi(x) \frac{d}{dx}\left(\frac{u'(x)}{\sqrt{1 + (u'(x))^2}}\right) dx = 0.$$
Now, since ϕ is an arbitrary function, we get the condition
$$\frac{d}{dx}\left(\frac{u'(x)}{\sqrt{1 + (u'(x))^2}}\right) = 0$$
and so
$$\frac{u'(x)}{\sqrt{1 + (u'(x))^2}} = \lambda,$$
where $|\lambda| < 1$. Therefore
$$\frac{(u'(x))^2}{1 + (u'(x))^2} = 1 - \frac{1}{1 + (u'(x))^2} = \lambda^2$$
and so
$$u'(x) = \sqrt{\frac{\lambda^2}{1 - \lambda^2}} \;\Rightarrow\; u(x) = x\frac{|\lambda|}{\sqrt{1 - \lambda^2}} + C.$$
Now we choose the constants λ and C so that $u(0) = a$ and $u(1) = b$.

G 2 Let

$$\mathcal{I}[u] = \int_0^1 \sqrt{1 + (u'(x))^2} + \lambda u(x) dx$$

for $u \in D$ with $\lambda \neq 0$. Write the Euler-Lagrange equations and solve them.

This is done similarly as above. We get the equation

$$-\frac{d}{dx}\left(\frac{u'(x)}{\sqrt{1 + (u'(x))^2}}\right) + \lambda = 0$$

and so

$$\frac{u'(x)}{\sqrt{1 + (u'(x))^2}} = \lambda x + C.$$

This gives

$$1 - \frac{1}{1 + (u'(x))^2} = (\lambda x + C)^2$$

and so

$$u'(x) = -\frac{\lambda x + C}{\sqrt{1 - (\lambda x + C)^2}},$$

where we assumed that $\lambda x + C \leq 0$. It is easy to integrate the above equation:

$$u(x) = \sqrt{1 - (\lambda x + C)^2} + C'.$$

This is a circle, where C and C' are chosen in such a way that $u \in D$.

G 3 Using Lagrange multipliers, minimize $f(x, y) = x^2 + y^2$ subject to the condition $x + y = 1$.

We write

$$F(x, y, \lambda) = x^2 + y^2 - \lambda(x + y - 1)$$

and the necessary conditions are

$$\begin{aligned} 0 =& F_x = 2x - \lambda \\ 0 =& F_y = 2y - \lambda \\ 0 =& F_\lambda = -x - y + 1. \end{aligned}$$

From this we compute:

$$\begin{aligned} x = y =& \frac{\lambda}{2} \\ \lambda =& 1 \end{aligned}$$

and so $x = y = \frac{1}{2}$. This in fact is a minimum, since the matrix of second derivatives

$$D^2F\left(\frac{1}{2}, \frac{1}{2}, 1\right) = \begin{pmatrix} 2 & 0 & -1 \\ 0 & 2 & -1 \\ -1 & -1 & 0 \end{pmatrix}$$

is positive-definite.

Hausübung

H 1 The isoperimetric problem

Let

$$D = \{u \in C^1[-1,1] \colon u(-1) = u(1) = 0\}$$

and let $\Omega_u = \{(x,y) \in [-1,1] \times \mathbb{R}_+ \colon 0 \le y \le u(x)\}$, $\Gamma_u = \{(x,y) \in [-1,1] \times \mathbb{R}_+ \colon y = u(x)\}$. Let $\mathcal{A}[u]$ denote the area of Ω_u and let $\mathcal{L}[u]$ be the length of the curve Γ_u. Our goal is to solve Euler-Lagrange equations for the problem

$$\mathcal{L}[u] = \min_{v \in D} \mathcal{L}[v] \quad \text{with the condition}$$
$$\mathcal{A}[u] = \pi,$$

that is, we want to find a curve of least possible length, starting at the point $(-1,0)$ and ending in $(1,0)$, with area under this curve being equal to π.

1. Write the length $\mathcal{L}[u]$ and area $\mathcal{A}[u]$ functionals.

$$\mathcal{L}[u] = \int_0^1 \sqrt{1 + (u'(x))^2} dx, \quad \mathcal{A}[u] = \int_0^1 u(x) dx.$$

2. Take two functions $v, w \in C_0^1[-1,1]$. Let $s, t \in \mathbb{R}$. Write the function $f(s,t) = \mathcal{L}[u+sv+tw]$ and compute its derivative. Find the conditions on s, t for which $\mathcal{A}[u+sv+tw] = \pi$.

$$\mathcal{L}[u + s\phi + t\psi] = \int_0^1 \sqrt{1 + (u'(x) + s\phi'(x) + t\psi'(x))^2} dx$$
$$f_s(s,t) = \int_0^1 \frac{\phi'(x)}{\sqrt{1 + (u'(x) + s\phi'(x) + t\psi'(x))^2}} dx$$
$$f_t(s,t) = \int_0^1 \frac{\psi'(x)}{\sqrt{1 + (u'(x) + s\phi'(x) + t\psi'(x))^2}} dx.$$

The conditions on s and t are

$$s \int_0^1 \psi(x) dx + t \int_0^1 \phi(x) dx = 0$$

(since it is supposed that $\mathcal{A}[u] = \pi$).

3. Write the Euler-Lagrange equations and solve them.

Our goal is to minimize (cf. **G 3***)*

$$F(s,t,\lambda) = \int_0^1 \sqrt{1 + (u'(x) + s\psi'(x) + t\phi'(x))^2} + \lambda(u(x) + s\psi(x) + t\phi(x) - \pi) dx.$$

The minimum should be attained at $s = t = 0$, since u is this minimum. We get two equations which in fact are the same:

$$-\frac{d}{dx}\left(\frac{1}{\sqrt{1 + (u'(x))^2}}\right) + \lambda = 0.$$

This has been solved in **G 2** *and the solution is a circle.*