Variationsrechnung

2. Übung, Lösungsvorschlag

Gruppenübung

G 1 a) *We have to prove that* $[uv]_{C^{0,\alpha}(D)}$ *is finite for every compact* $D \subset \Omega$*. Take* $x, y \in D$ *such that* $x \neq y$.

$$\begin{aligned}\frac{|u(x)v(x) - u(y)v(y)|}{|x-y|^\alpha} &= \frac{|u(x)v(x) - u(x)v(y) + u(x)v(y) - u(y)v(y)|}{|x-y|^\alpha} \\ &\leq |u(x)|\frac{|v(x)-v(y)|}{|x-y|^\alpha} + |v(y)|\frac{|u(x)-u(y)|}{|x-y|^\alpha}. \qquad (\star)\end{aligned}$$

The functions u, v *are continuous on a compact set* D*, therefore they are bounded. The difference quotients are also bounded, from the assumptions on* u, v*. Therefore the right-hand side of* $(\star)$ *is bounded, and so is its supremum.*

b) *It is sufficient to prove the inclusions for* $k = 0$*. Also, the inclusion* $C^{0,1}(\overline{\Omega}) \subset C^{0,\beta}(\overline{\Omega})$ *is a consequence of the inclusion* $C^{0,\beta}(\overline{\Omega}) \subset C^{0,\alpha}(\overline{\Omega})$ *for* $0 < \alpha \leq \beta \leq 1$*. To prove it, take* $x, y \in \Omega$ *such that* $x \neq y$*. Suppose* $|x-y| \leq 1$*. Then* $|x-y|^\beta \geq |x-y|^\alpha$ *and so*

$$\frac{1}{|x-y|^\beta} \leq \frac{1}{|x-y|^\alpha}.$$

This gives

$$\frac{|u(x)-u(y)|}{|x-y|^\beta} \leq \frac{|u(x)-u(y)|}{|x-y|^\alpha} \leq C \qquad (\dagger)$$

for $|x-y| \leq 1$*. For* $|x-y| \geq 1$ *we have* $|x-y|^\alpha \geq 1$ *and so*

$$\frac{1}{|x-y|^\alpha} \leq 1$$

which gives

$$\frac{|u(x)-u(y)|}{|x-y|^\alpha} \leq |u(x)-u(y)| \leq 2\sup_{x\in\Omega}|u(x)| \leq C \qquad (\ddagger)$$

because $u \in C(\overline{\Omega})$.

Combining $(\dagger)$ *and* $(\ddagger)$ *we get the desired inclusion. The remaining inclusion is obvious from definition of* $C^{0,\alpha}$.

We will prove $C^1(\Omega) \subset C^{0,1}(\Omega)$*. Take a compact set* $D \subset \Omega$ *and* $x \neq y \in D$*. If* $u \in C^1(\Omega)$ *then the mean-value theorem holds:*

$$|u(x)-u(y)| \leq \sup_{\xi\in D}|\nabla u(\xi)||x-y|.$$

Since ∇u *is continuous on* Ω *and* D *is a compact set, the supremum on the right is finite. Dividing by* $|x-y|$ *we get boundedness of* $[u]_{C^{0,1}(D)}$ *for any* D*, which proves that* $u \in C^{0,1}(\Omega)$.