Fachbereich Mathematik
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Variationsrechnung

4. Übung

Gruppenübung

G 1 Let $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$ for $1 \le p \le \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Define the *convolution* of f and g by the formula

$$(f \star g)(x) = \int_{\mathbb{R}^n} f(x-y)g(y)dy.$$

1. Prove that $f \star g = g \star f$.
2. For $f = \chi_{[0,1]}$ compute $f \star f$ and sketch its graph.

G 2 Let $1 \le p < \infty$.

1. Prove that the space $C_0(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$.
   Hint: Start by approximating characteristic functions by continuous functions.
2. Prove that for $f \in L^p(\mathbb{R}^n)$

   $$\int_{\mathbb{R}^n} |f(x+y) - f(x)|^p dx \to 0$$

   when $|y| \to 0$.

Hausübung

H 1 Let $\rho \in C_0^\infty(\mathbb{R}^n)$ be such that

$$\rho(x) \ge 0, \quad \int_{\mathbb{R}^n} \rho(x)dx = 1$$

and take $\rho_\epsilon(x) = C_\epsilon \rho\left(\frac{x}{\epsilon}\right)$ for $\epsilon > 0$.

1. Compute C_ϵ for which $\int_{\mathbb{R}^n} \rho_\epsilon(x)dx = 1$.
2. Take $1 \le p < \infty$. Prove that

   $$f \star \rho_\epsilon \to f \quad \text{in } L^p(\mathbb{R}^n)$$

   when $\epsilon \to 0$ and from this conclude that the space $C_0^\infty(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$.
3. Show that the above convergence does not hold for $p = \infty$.