Fachbereich Mathematik
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Variationsrechnung

3. Übung

Gruppenübung

G 1 Consider the functional

$$\mathcal{I}[u] = \int_0^1 \left(|u'(s)| - 1\right)^2 + |u(s)|^2 ds.$$

1. Prove that the function $F(u, \xi) = (|\xi| - 1)^2 + u^2$ with $(u, \xi) \in \mathbb{R}^2$ is not convex.
2. A function f is *piecewise continuously differentiable* on $[0, 1]$ if $f \in C([0, 1])$ and there exist $0 = a_0 < a_1 < ... < a_n = 1$ such that $u \in C^1(\bigcup_{i=1}^n (a_{i-1}, a_i))$. Find a sequence of piecewise continuously differentiable functions u_n such that $\mathcal{I}[u_n] \to 0$. Let $u_n \to u$ pointwise in $[0, 1]$. Is it true that $\lim_{n\to\infty} \mathcal{I}[u_n] = \mathcal{I}[u]$?

G 2 Prove that a continuous function $f\colon [a, b] \to \mathbb{R}$ is convex if and only if

$$\frac{f(x) + f(y)}{2} \geq f\left(\frac{x + y}{2}\right)$$

for all $x, y \in [a, b]$.

Hausübung

H 1 Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be convex.

1. Suppose that f has a local minimum at x, i.e. there exists an open neighborhood U of x such that for all $y \in U$ $f(x) \leq f(y)$. Prove that then x is a global minimum, i.e. $f(x) \leq f(y)$ for all $x \in \mathbb{R}^n$.
2. Prove that f is *hemicontinuous*, i.e. continuous along straight lines.
3. Suppose f is continuously differentiable.
   (a) Prove that f is convex if and only if

   $$f(y) \geq f(x) + \nabla f(x) \cdot (y - x)$$

   for all $x, y \in \mathbb{R}^n$.
   (b) Prove that f is convex if and only if its gradient ∇f is *monotone*, i.e.

   $$(\nabla f(x) - \nabla f(y)) \cdot (x - y) \geq 0$$

   (cf. script, p. 36).
4. Prove that if both f and $-f$ are convex then f is affine, i.e. $f(x) = Ax + b$ for some $(n \times n)$ matrix A and vector $b \in \mathbb{R}^n$.