Fachbereich Mathematik
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Variationsrechnung

2. Übung

Gruppenübung

G 1 Let
$$D = \{u \in C^1[0,1] \colon u(0) = a \wedge u(1) = b\}$$
and let $\mathcal{I}[u]$ denote the length of curve $x \mapsto (x, u(x))$ for $u \in D$.

1. Write $\mathcal{I}[u]$ in integral form.
2. Let $\phi \in C^0_\infty(0,1)$ and let $f(t) = \mathcal{I}[u + t\phi]$ for $t \in \mathbb{R}$. Observe that $u + t\phi \in D$ and then compute $f'(t)$.
3. Suppose $u \in D$ is such that
$$\mathcal{I}[u] = \min_{v \in D} \mathcal{I}[v].$$
What condition must f satisfy? Is this condition also sufficient? Write a proper differential equation for u and solve it.

G 2 Let
$$\mathcal{I}[u] = \int_0^1 \sqrt{1 + (u'(x))^2} + \lambda u(x) dx$$
for $u \in D$ with $\lambda \neq 0$. Write the Euler-Lagrange equations and solve them.

G 3 Using Lagrange multipliers, minimize $f(x,y) = x^2 + y^2$ subject to the condition $x + y = 1$.

Hausübung

H 1 **The isoperimetric problem**
Let
$$D = \{u \in C^1[-1,1] \colon u(-1) = u(1) = 0\}$$
and let $\Omega_u = \{(x,y) \in [-1,1] \times \mathbb{R}_+ \colon 0 \leq y \leq u(x)\}$, $\Gamma_u = \{(x,y) \in [-1,1] \times \mathbb{R}_+ \colon y = u(x)\}$. Let $\mathcal{A}[u]$ denote the area of Ω_u and let $\mathcal{L}[u]$ be the length of the curve Γ_u. Our goal is to solve Euler-Lagrange equations for the problem
$$\begin{aligned} \mathcal{L}[u] &= \min_{v \in D} \mathcal{L}[v] \quad \text{with the condition} \\ \mathcal{A}[u] &= \pi, \end{aligned}$$
that is, we want to find a curve of least possible length, starting at the point $(-1,0)$ and ending in $(1,0)$, with area under this curve being equal to π.

1. Write the length $\mathcal{L}[u]$ and area $\mathcal{A}[u]$ functionals.
2. Take two functions $v, w \in C^1_0[-1,1]$. Let $s,t \in \mathbb{R}$. Write the function $f(s,t) = \mathcal{L}[u+sv+tw]$ and compute its derivative. Find the conditions on s,t for which $\mathcal{A}[u+sv+tw] = \pi$.
3. Write the Euler-Lagrange equations and solve them.