Fachbereich Mathematik
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Introduction to the calculus of variations
Exercises No. 1

P1: Let $\Omega \subseteq \mathbb{R}^n$ be an open set and let $k \in \mathbb{N}_0$, $0 < \alpha \le 1$. Let

$$C^{0,\alpha}(\Omega) = \{u \in C(\Omega) \mid [u]_{C^{0,\alpha}(D)} < \infty \text{ for every compact set } D \subseteq \Omega\},$$

where

$$[u]_{C^{0,\alpha}(D)} = \sup_{\substack{x,y \in D \\ x \neq y}} \left\{ \frac{|u(x)-u(y)|}{|x-y|^\alpha} \right\},$$

$$\begin{aligned} C^{0,\alpha}(\overline{\Omega}) &= \{u \in C(\overline{\Omega}) \mid [u]_{C^{0,\alpha}(\Omega)} < \infty\}, \\ C^{k,\alpha}(\Omega) &= \{u \in C^k(\Omega) \mid \nabla^k u \in C^{0,\alpha}(\Omega)\}, \\ C^{k,\alpha}(\overline{\Omega}) &= \{u \in C^k(\overline{\Omega}) \mid [\nabla^k u]_{C^{0,\alpha}} < \infty\}. \end{aligned}$$

The space $C^{k,\alpha}(\overline{\Omega})$ is a normed space with norm

$$\|u\|_{C^{k,\alpha}} = \|u\|_{C^k} + [\nabla^k u]_{C^{0,\alpha}} .$$

Prove the following statements:

a) If $u, v \in C^{0,\alpha}(\Omega)$, then $uv \in C^{0,\alpha}(\Omega)$.

b) If $0 < \alpha \le \beta \le 1$ and $k \ge 0$, then

$$C^k(\overline{\Omega}) \supseteq C^{k,\alpha}(\overline{\Omega}) \supseteq C^{k,\beta}(\overline{\Omega}) \supseteq C^{k,1}(\overline{\Omega}).$$

c) If Ω is convex and bounded, then

$$C^{k,1}(\overline{\Omega}) \supseteq C^{k+1}(\Omega).$$

P2: Let $\Omega \subseteq \mathbb{R}^n$ be open and let $1 \le p < \infty$. Let

$$L^p(\Omega) = \{f : \Omega \to \mathbb{R} \mid |f|^p \text{ is integrabel on } \Omega\}.$$

This is a normed space with the norm

$$\|f\|_p = \Big(\int_\Omega |f(x)|^p \, dx \Big)^{1/p}.$$

Now let $\Omega = (0,1)$ and $\alpha > 0$. Let the sequence $\{f_\ell\}_{\ell=1}^\infty$ be defined by

$$f_\ell(x) = \begin{cases} \ell^\alpha, & x \in \big(0, \frac{1}{\ell}\big) \\ 0, & x \in \big[\frac{1}{\ell}, 1\big). \end{cases}$$

Prove that

$$\lim_{\ell \to \infty} f_\ell \text{ in } L^p(\Omega) \iff 0 < \alpha < \frac{1}{p}.$$