Partielle Differentialgleichungen

9. Übung
Lösungsvorschlag

Gruppenübung

G 1 **[Umgekehrter Mittelwertsatz]** $u \in C^2(U)$ erfülle die Mittelwertformel, d.h. für alle $B(x,r) \subset U$ gelte

$$u(x) = \fint_{\partial B(x,r)} u(y) dS_y.$$

Zeigen Sie, dass u harmonisch in U ist.
Hinweis: $\frac{d}{dr} \fint_{\partial B(x,r)} u(y) dS_y = 0$.

The hint follows from the fact that the left-hand side of the above equality is independent of r and so the right-hand side must be constant, as a function of r. Just like in the script we compute, using the transformation of variables $y = x + rz$

$$\begin{aligned} \fint_{\partial B(x,r)} u(y) dS_y =& \frac{1}{|\partial B(x,r)|} \int_{\partial B(x,r)} u(y) dS_y = \frac{1}{\omega_n r^{n-1}} \int_{\partial B(x,r)} u(y) dS_y \\ =& \frac{1}{\omega_n r^{n-1}} \int_{\partial B(0,1)} u(x+rz) r^{n-1} dS_z = \frac{1}{\omega_n} \int_{\partial B(0,1)} u(x+rz) dS_z \end{aligned}$$

because $dS_y = r^{n-1} dS_z$. Therefore

$$\begin{aligned} \frac{d}{dr} \fint_{\partial B(x,r)} u(y) dS_y =& \frac{d}{dr} \frac{1}{\omega_n} \int_{\partial B(0,1)} u(x+rz) dS_z = \frac{1}{\omega_n} \int_{\partial B(0,1)} z \cdot \nabla u(x+rz) dS_z \\ =& \frac{1}{\omega_n} \int_{\partial B(0,1)} \frac{\partial u}{\partial \nu}(x+rz) dS_z = \frac{1}{\omega_n r^{n-1}} \int_{\partial B(x,r)} \frac{\partial u}{\partial \nu}(y) dS_y. \end{aligned}$$

This is because, with the aid of chain rule, we have

$$\begin{aligned} \frac{d}{dr} u(x+rz) =& \frac{d}{dr} u(x_1 + rz_1, x_2 + rz_2, ..., x_n + rz_n) = z_1 \frac{\partial u}{\partial x_1} + z_2 \frac{\partial u}{\partial x_2} + ... + z_n \frac{\partial u}{\partial x_n} \\ =& (z_1, z_2, ..., z_n) \cdot \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, ..., \frac{\partial u}{\partial x_n} \right) = z \cdot \nabla u(x+rz). \end{aligned}$$

Now Gauss theorem states that

$$\int_{\partial B(x,r)} \frac{\partial u}{\partial \nu}(y) dS_y = \int_{B(x,r)} \Delta u(y) dy$$

and so we obtain the equality

$$\int_{B(x,r)} u(y) dy = 0 \qquad (\star)$$

for every $x \in U$ and every $r > 0$ such that $B(x,r) \subset U$. From this it follows that $\Delta u = 0$ in U.

In fact, suppose on the contrary that $\Delta u(x_0) = \alpha \neq 0$ at some point $x_0 \in U$. Suppose that $\alpha > 0$ (the case $\alpha < 0$ is treated similarly). Then, since $u \in C^2$, the function Δu is continuous and therefore there exists a small radius $r > 0$ such that on the ball $B(x_0, r)$ we have $\Delta u > \frac{\alpha}{2}$. But then, we have

$$\int_{B(x_0,r)} u(y) dy > \frac{\alpha}{2} |B(x_0, r)| > 0,$$

which contradicts $(\star)$.

G 2 [**Schwarz Symmetrieprinzip**] Sei $U \subset \mathbb{R}^n$ offen, symmetrisch bezüglich der Hyperebene $\{x_n = 0\}$. Sei $U_+ = \{x_n > 0\} \cap U$ und $u\colon U_+ \to \mathbb{R}$ harmonisch, stetig bis zum Rand und sei

$$\hat{u}(x) = \begin{cases} u(x), & \text{für } x \in U_+ \\ -u(S(x)), & \text{für } x \in U_- = U \setminus \overline{U_+}, \end{cases}$$

wobei $S(x_1, ..., x_n) = (x_1, ..., x_{n-1}, -x_n)$ die Symmetrietransformation bezüglich $\{x_n = 0\}$ ist. Zeigen Sie, dass wenn $u|_{\{x_n=0\}} = 0$ gilt, dann $\hat{u}$ harmonisch in U ist.

It is easy to see, by straightforward computation, that $\hat{u}$ is harmonic in the sets U_+ and U_-. The problem is to prove the harmonicity of $\hat{u}$ on the hyperplane $x_n = 0$. For this we use the exercise **G 1**. *In fact, it is sufficient to prove that*

$$\hat{u}(x_1, ..., x_{n-1}, 0) = \frac{1}{\omega r^{n-1}} \int_{\partial B(x,r)} \hat{u}(y) dS_y$$

which is equivalent to

$$0 = \int_{\partial B(x,r)} \hat{u}(y) dS_y \qquad (\dagger)$$

for $x = (x_1, ..., x_{n-1}, 0)$ and all $r > 0$ such that $B(x, r) \subset U$. But $(\dagger)$ follows easily from the definition of $\hat{u}$, because the sphere $\partial B(x, r)$ is symmetric with respect to the hyperplane $x_n = 0$ and the function $\hat{u}$ is odd (ungerade Funktion) with respect to the x_n variable.

G 3 [**Satz von Liouville**] Sei u

1. harmonisch in $\mathbb{R}^n$
2. harmonisch in $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n \colon x_n > 0\}$, $u_{\{x_n=0\}} = 0$

und beschränkt. Zeigen Sie, dass u konstant ist.
Hinweis: Nutzen Sie die Mittelwertformel.

The second part follows from the first one and the Schwartz symmetry principle (in fact, the constant must be equal to 0).

For the first part we apply the inverse middle value theorem from **H 2** *(this is different from* **G 1** *only by the fact that we take balls $B(x, r)$ instead of spheres $\partial B(x, r)$). It is sufficient to prove that*

$$u(x) - u(y) = 0$$

for all $x, y \in \mathbb{R}^n$. Take $r > 0$ large enough and write

$$\begin{aligned} u(x) - u(y) =& \frac{1}{|B(x,r)|} \int_{B(x,r)} u(y) dy - \frac{1}{|B(y,r)|} \int_{B(y,r)} u(y) dy \\ =& \frac{1}{|B_r|} \left(\int_{B(x,r)} u(y) dy - \int_{B(y,r)} u(y) dy \right) \end{aligned}$$

since $|B(x, r)| = |B(y, r)| = |B_r|$. Let $A \Delta B = (A \setminus B) \cup (B \setminus A)$ be the symmetric difference of sets A and B. Then

$$|u(x) - u(y)| \leq \frac{1}{|B_r|} \int_{B(x,r) \Delta B(y,r)} |u(y)| dy.$$

But u is assumed to be bounded: $|u(y)| \leq M$ for all $y \in \mathbb{R}^n$. Thus

$$|u(x) - u(y)| \leq M \frac{|B(x, r) \Delta B(y, r)|}{|B_r|}. \qquad (\circ)$$

It is left as an exercise (geometric) to prove that the right-hand side of $(\circ)$ goes to zero when $r \to \infty$.