Partielle Differentialgleichungen

8. Übung
Lösungsvorschlag

Hausübung

H 1 Let $u(x,y) = \ln\sqrt{x^2+y^2}$ for $U = \mathbb{R}^2 \setminus \{(0,0)\}$.

1. Prove that $\Delta u(x,y) = 0$ for $(x,y) \in U$.

   In polar coordinates:

$$\Delta u = (\ln r)_{rr} + \frac{1}{r}(\ln r)_r = -\frac{1}{r^2} + \frac{1}{r^2} = 0.$$

2. For $r > 0$ calculate the integral

$$\int_{\partial B(0,r)} \frac{\partial u}{\partial \nu}(y) dS_y.$$

   Again, in polar coordinates $\frac{\partial u}{\partial \nu} = u_r$ *and so*

$$\int_{\partial B(0,r)} u_r dS = \int_{\partial B(0,r)} \ln r dS = |\partial B(0,r)| \ln r.$$

3. Explain why the above result doesn't contradict Gauss's theorem.

   The function u *is defined in the set* $U = B(0,r) \setminus \{(0,0)\}$ *and so* $U = \partial B(0,r) \cup \{(0,0)\}$ *and in fact, we did not integrate at* $(0,0)$*, where the function is infinite (*Δu *looks at (0, 0) similarly to Dirac's delta).*

H 2 Prove that $\Delta u = v_{rr} + \frac{1}{r}v_r + \frac{1}{r^2}v_{\phi\phi}$ in radial coordinates.
Hint: Let

$$x = r\cos\phi, \quad y = r\sin\phi.$$

Take $u(x,y) = v(r,\phi) = v\left(\sqrt{x^2+y^2}, \arctan\frac{y}{x}\right)$. Then compute $\partial_x u$, $\partial_{xx} u$, $\partial_y u$, $\partial_{yy} u$ using the chain rule.

Similar exercises have been done before.