Partielle Differentialgleichungen

7. Übung
Lösungsvorschlag

Gruppenübung

G 1 Let $u \in C^3(\Omega)$ be a harmonic function in an open set $\Omega \in \mathbb{R}^n$, i.e.

$$\Delta u(x) = 0 \quad \text{for } x \in \Omega.$$

1. Prove that $w(x) = x \cdot \nabla u(x)$ is also harmonic in Ω.

   We have
   $$w(x) = x \cdot \nabla u(x) = (x_1, ..., x_n) \cdot (\partial_1 u, ..., \partial_n u) = \sum_{i=1}^{n} x_i \partial_i u.$$
   Now
   $$\partial_k w = \partial_k \left(\sum_{i=1}^{n} x_i \partial_i u \right) = \sum_{i=1}^{n} \partial_k (x_i \partial_i u) = \partial_k u + \sum_{i=1}^{n} x_i \partial_{ik}^2 u$$
   since $\partial_k x_i = \delta_i^k$ (the Kronecker delta). Similarly
   $$\begin{aligned}\partial_k^2 w =& \partial_k(\partial_k w) = \partial_k \left(\partial_k u + \sum_{i=1}^{n} x_i \partial_{ik}^2 u \right) = \partial_{kk}^2 u + \sum_{i=1}^{n} \partial_k \left(x_i \partial_{ik}^2 u \right) \\ =& \partial_{kk}^2 u + \partial_{kk}^2 + \sum_{i=1}^{n} x_i \partial_{ikk}^3 u = 2\partial_{kk}^2 + \sum_{i=1}^{n} x_i \partial_{ikk}^3 u.\end{aligned}$$
   Therefore
   $$\begin{aligned}\Delta w =& \sum_{k=1}^{n} \partial_{kk}^2 w = 2 \sum_{k=1}^{n} \partial_{kk}^2 u + \sum_{k=1}^{n} \sum_{i=1}^{n} x_i \partial_{ikk}^3 u \\ =& 2\Delta u + \sum_{i=1}^{n} x_i \partial_i \left(\sum_{k=1}^{n} \partial_{kk}^2 u \right) = 2\Delta u + \sum_{i=1}^{n} x_i \partial_i \left(\Delta u \right) = 0.\end{aligned}$$

2. *Take $w = u^2$. Then*
   $$\partial_k w = \partial_k \left(u^2 \right) = 2u\partial_k u.$$
   Also
   $$\partial_{kk}^2 w = 2\partial_k (u \partial_k u) = 2u\partial_{kk}^2 u + 2(\partial_k u)^2.$$
   Therefore
   $$\Delta w = 2u\Delta u + 2|\nabla u|^2 = 2|\nabla u|^2 \geq 0.$$
   Now take $w = e^u$. We have
   $$\partial_k w = e^u \partial_k u$$
   and
   $$\partial_{kk}^2 w = e^u (\partial_k u)^2 + e^u \partial_{kk}^2 u$$
   which gives
   $$\Delta w = e^u |\nabla u|^2 + e^u \Delta u = e^u |\nabla u|^2 \geq 0.$$

G 2 Let $f\colon D \to \mathbb{C}$ be a holomorphic function, with $D \subset \mathbb{C}$ and let $f(x,y) = u(x,y) + iv(x,y)$. Then u and v satisfy the *Cauchy-Riemann equations*:

$$u_x = v_y, \quad u_y = -v_x.$$

1. $$u_{xx} = v_{yx} = v_{xy} = -u_{yy} \quad \Rightarrow \quad u_{xx} + u_{yy} = 0$$

   and similarly for v.

2. *We have:*

   $$v_x = -u_y = -3xy^2 + x^3$$

   and so, after integration with respect to x

   $$v = -\frac{3}{2}x^2y^2 + \frac{1}{4}x^4 + \phi(y),$$

   where the function ϕ is to be determined. Now we have:

   $$-3x^2y + \phi'(y) = v_y = u_x = y^3 - 3x^2y$$

   and so

   $$\phi'(y) = y^3 \quad \Rightarrow \quad \phi(y) = \frac{1}{4}y^4 + C$$

   with C being a constant. Therefore

   $$u(x,y) = \frac{1}{4}x^4 - \frac{3}{2}x^2y^2 + \frac{1}{4}y^4 + C.$$

3. *Since integrating and derivating a polynomial gives another polynomial, it is not difficult to observe from the procedure above that v must be a polynomial if u is a polynomial.*