Partielle Differentialgleichungen

6. Übung
Lösungsvorschlag

Gruppenübung

G 1 1.

$$\begin{aligned}\rho\partial_t u + k\rho^\alpha \nabla\rho =& (\rho_0+\rho_1)\partial_t u + k(\rho_0+\rho_1)^\alpha \nabla(\rho_0+\rho_1)\\ =& \rho_0\partial_t u + k\rho_0^\alpha\nabla\rho_1 + s \approx \rho_0\partial_t u + k\rho_0^\alpha \nabla\rho_1\end{aligned}$$

where s are small terms. For the second equation:

$$\partial_r\rho + \operatorname{div}(\rho u) = \partial_t(\rho_0+\rho_1) + \operatorname{div}(\rho_0 u + \rho_1 u) = \partial_t\rho_1 + \rho_0 \operatorname{div} u + s \approx \partial_t\rho_1 + \rho_0 \operatorname{div} u.$$

2.

$$\partial_t^2\rho_1 = \partial_t(-\rho_0 \operatorname{div} u) = -\operatorname{div}(\rho_0\partial_t u) = k\rho_0^\alpha \operatorname{div}\nabla\rho_1 = c^2\Delta\rho_1$$

with $c^2 = k\rho_0^\alpha$.

G 2 *Kirchoff's formula (in 3 dimensions) is*

$$u(t,x) = \frac{1}{4\pi c^2 t}\inf_{\partial B_{ct}(x)} u_1(y)dS_y + \partial_t\left(\frac{1}{4\pi c^2 t}\int_{\partial B_{ct}(x)} u_0(y)dS_y\right).$$

For $x=0$, since $u_1\equiv 0$, we thus have

$$\begin{aligned}u(t,0) =& \partial_t\left(\frac{1}{4\pi c^2 t}\int_{\partial B_{ct}(0)}\frac{1}{1+|y|^2}dS_y\right) = \partial_t\left(\frac{1}{4\pi c^2 t}\cdot\frac{1}{1+c^2t^2}\cdot 4\pi c^2 t^2\right)\\ =& \partial_t\left(\frac{t}{1+c^2t^2}\right) = \frac{1-c^2t^2}{(1+c^2t^2)^2}.\end{aligned}$$

G 3 1. *Integrate the equation over $x\in\Omega$ to obtain*

$$\int_\Omega \Delta u_{tt}(t,x)dx = \int_\Omega \Delta u(t,x) + \int_\Omega f(t,x) = \int_{\partial\Omega}\frac{\partial u}{\partial\nu}(x)dS_x = 0.$$

Therefore

$$\frac{d}{dt}\int_\Omega u_t(t,x)dx = 0$$

and so

$$\int_\Omega u_t(t,x)dx = c = \int_\Omega u(t^\star,x)dx = 0.$$

2. *From the above we have*

$$0 = \int_\Omega u_t(t,x)dx = \frac{d}{dt}\int_\Omega u(t,x)dx$$

and so

$$\int_\Omega u(t,x)dx = c = \int_\Omega u(t^{\star\star},x)dx = 0.$$