Partielle Differentialgleichungen

5. Übung
Lösungsvorschlag

Hausübung

H 1 Let u be the solution to the initial-boundary problem

$$\begin{aligned} u_{tt} =& c^2\Delta u + f \quad \text{for } (t,x) \in \mathbb{R}_+ \times \Omega \\ u(t,x) =& 0 \quad \text{for } (t,x) \in \mathbb{R}_+ \times \partial\Omega \\ u(0,x) =& u^0(x) \quad \text{for } x \in \Omega \\ u_t(0,x) =& u^1(x) \quad \text{for } x \in \Omega, \end{aligned}$$

where $\Omega \subset \mathbb{R}^n$ is open and bounded. Let

$$E(t) = \int_\Omega \left(u_t^2 + c^2|\nabla u|^2\right) dx$$

be the energy at point $t \in \mathbb{R}_+$.

1. Prove that if $f = 0$ then $E(t)$ is a constant function.
   Hint: Compute the derivative and use integration by parts.

   Computing the derivative we get:

$$\begin{aligned} \frac{d}{dt}E(t) =& \int_\Omega \left(2u_t u_{tt} + 2c^2 \nabla u \cdot \nabla u_t\right) dx \\ =& 2\int_\Omega \left(u_t u_{tt} - c^2 \Delta u u_t\right) dx + 2\int_{\partial\Omega} c^2 \frac{\partial u}{\partial \nu} u_t dx = 0. \end{aligned}$$

   Therefore $E(t) = E(0) = const$.

2. Prove that the solution with arbitrary f is unique.
   Hint: Assume there are two solutions u and v. What initial-boundary problem does the difference $u - v$ satisfy?

   Take $w = u - v$, then w satisfies the wave equation with $f = 0$ and zero initial values. This means that the energy is constant and $E(0) = 0$ and therefore $w_t, \nabla w = 0$ and so $w(t,x) = const$ and again taking into account that the initial conditions are zero, $w(t,x) = 0$.

H 2 Let $u \in C^2(\mathbb{R}^3)$ be a solution of the partial differential equation

$$-\Delta u(x) = g(x), \quad x \in \mathbb{R}^3.$$

1. Prove that

$$\frac{1}{r^2}\int_{B_r(x)} \Delta u(y) dy = \frac{d}{dr}\left(\frac{1}{r^2}\int_{\partial B_r(x)} u(y) dS_y\right).$$

   We have

$$\int_{B_r(x)} \Delta u(y) dy = \int_{\partial B_r(x)} \nabla u(y) \cdot \nu dS_y = \int_{\partial B_1(0)} \nabla u(x + rz) \cdot z dS_z$$

   and also

$$\begin{aligned} \frac{d}{dr}\left(\frac{1}{r^2}\int_{\partial B_r(x)} u(y) dS_y\right) =& \frac{d}{dr}\left(\frac{1}{r^2}\int_{\partial B_1(0)} r^2 u(x + rz) dS_z\right) \\ =& \int_{\partial B_1(0)} \nabla u(x + rz) \cdot z dS_z \end{aligned}$$

   and from this follows the desired equality.

2. Use this equation to show that

$$u(x) = \frac{1}{\omega_3 r^2} \int_{\partial B_r(x)} u(y) dS_y + \int_0^r \left(\frac{1}{\omega_3 r^2} \int_{B_r(x)} g(y) dy \right) dr.$$

We have

$$-\frac{1}{\omega_3 r^2} \int_{B_r(x)} g(y) dy = \frac{1}{\omega_3 r^2} \int_{B_r(x)} \Delta u(y) dy = \frac{d}{dr} \left(\frac{1}{\omega_3 r^2} \int_{\partial B_r(x)} u(y) dS_y \right)$$

and so, after integration with respect to r:

$$\frac{1}{\omega_3 r^2} \int_{\partial B_r(x)} u(y) dS_y + \int_0^r \left(\frac{1}{\omega_3 r^2} \int_{B_r(x)} g(y) dy \right) dr = \lim_{\epsilon \to 0^+} \int_{\partial B_\epsilon(x)} u(y) dS_y + C = u(x) + C$$

from the mean value theorem. Taking $r \to 0^+$ in the above equality we get

$$u(x) + 0 = u(x) + C$$

and so $C = 0$, which proves the desired result (notice that $G(r) = \frac{1}{\omega_3 r^2} \int_{B_r(x)} g(y) dy \to g(x)$ when $r \to 0$ and so $\int_0^r G(r) dr \to 0$ with $r \to 0$, when g is a continuous function).