# Partielle Differentialgleichungen

## 4. Übungen
### Lösungsvorschlag

**Gruppenübung**

**G 1** 1. *Since $a$, $b$ and $c$, $d$ lie on the same line parallel to $x = t$, then*

$$
\begin{aligned}
a^x - a^t =& b^x - b^t \\
c^x - c^t =& d^x - d^t.
\end{aligned}
$$

*Similarly*

$$
\begin{aligned}
a^x + a^t =& d^x + d^t \\
b^x + b^t =& c^x + c^t.
\end{aligned}
$$

*Therefore*

$$
\begin{aligned}
u(a) + u(c) =& \phi(a^x + a^t) + \psi(a^x - a^t) + \phi(c^x + c^t) + \psi(c^x - c^t) \\
=& \phi(d^x + d^t) + \psi(b^x - b^t) + \phi(b^x + b^t) + \psi(d^x - d^t) \\
=& u(b) + u(d).
\end{aligned}
$$

2. *Suppose there are two solutions: $u$ and $v$ and let $w = u - v$. Then the boundary and initial conditions for $w$ vanish. Take any point inside the triange bounded by $t = 0$, $x = t$, $x = l - t$. Then $w$ at this point is zero, from the d'Alambert equation. Next, choose the triangle bounded by $x = t$, $x = 0$, $x = l - t$. The solution at point inside this triangle is zero, since we can draw a rectangle, whose 3 vertices touch the boundaries at which we know the solution is zero and then use the formula from* **G1** *1. Similarly we extend this reasoning with triangless to the whole strip $[0, l] \times \mathbb{R}_+$.*
   *The functions should be of class $C^2$ and satisfy the conditions*

$$
\begin{aligned}
\alpha(0) =& g(0), \beta(0) = g(l) \\
\alpha'(0) =& h(0), \beta'(0) = h(l) \\
\alpha''(0) =& g'', \beta''(0) = g''(l)
\end{aligned}
$$

   *(the last equations follow from $u_{tt} = u_{xx}$).*

3. *Take any point $(\bar{x}, \bar{t}) \in \mathbb{R}^2$. The conditions are given along the lines $x = t + 1$ and $x = -t + 5$, their common point is $(x, t) = (3, 2)$. The lines parallel to these, passing through $(\bar{x}, \bar{t})$ are $x = t + \bar{x} - \bar{t}$ and $x = -t + \bar{x} + \bar{t}$. The points of crossing are $\left(\frac{1}{2}[\bar{x} + \bar{t} + 1], \frac{1}{2}[\bar{x} + \bar{t} - 1]\right)$ and $\left(\frac{1}{2}[\bar{x} - \bar{t} + 5], \frac{1}{2}[\bar{t} - \bar{x} + 5]\right)$. Therefore*

$$
\begin{aligned}
u(\bar{x}, \bar{t}) =& u\left(\frac{1}{2}[\bar{x} + \bar{t} + 1], \frac{1}{2}[\bar{x} + \bar{t} - 1]\right) + u\left(\frac{1}{2}[\bar{x} - \bar{t} + 5], \frac{1}{2}[\bar{t} - \bar{x} + 5]\right) - u(3, 2) \\
=& \alpha\left(\frac{1}{2}[\bar{x} + \bar{t} - 1]\right) + \beta\left(\frac{1}{2}[\bar{t} - \bar{x} + 5]\right) - u(3, 2).
\end{aligned}
$$

   *The obvious condition to satisfy is $u(3, 2) = \alpha(2) = \beta(2)$. Moreover, $\alpha, \beta \in C^2(\mathbb{R})$. Apart from that the data can be arbitrary.*