Partielle Differentialgleichungen

3. Übung
Lösungsvorschlag

Gruppenübung

G 1 1. *We use Schwarz' rule:*

$$\begin{aligned} v_{tt} =& u_{xtt} + u_{ttt} \\ =& u_{xxx} + u_{txx} = v_{xx}. \end{aligned}$$

2.

$$\begin{aligned} v_t =& 2u_x u_{xt} + 2u_t u_{tt} \\ v_{tt} =& 2u_{xt}^2 + 2u_x u_{xtt} + 2u_{tt}^2 + u_t u_{ttt} \\ v_x =& 2u_x u_{xx} + 2u_t u_{tx} \\ v_{xx} =& 2u_{xx}^2 + 2u_x u_{xxx} + 2u_{tx}^2 + 2u_t u_{txx} \end{aligned}$$

and so again $v_{tt} = v_{xx}$.

G 2 1. *We have*

$$u(t,x) = \hat{u}(\xi,\eta) = \hat{u}(\alpha x + \beta t, \gamma x + \delta t)$$

and so, using chain rule of differentiation

$$\begin{aligned} u_x =& \alpha \hat{u}_\xi + \gamma \hat{u}_\eta \\ u_{xx} =& \alpha^2 \hat{u}_{\xi\xi} + 2\alpha\gamma \hat{u}_{\xi\eta} + \gamma^2 \hat{u}_{\eta\eta} \\ u_{tt} =& \beta^2 \hat{u}_{\xi\xi} + 2\beta\delta \hat{u}_{\xi\eta} + \delta^2 \hat{u}_{\eta\eta} \end{aligned}$$

Plugging these into the wave equation we get

$$(c^2\alpha^2 - \beta^2)\hat{u}_{\xi\xi} + 2(c^2\alpha\gamma - \beta\delta)\hat{u}_{\xi\eta} + (c^2\gamma^2 - \delta^2)\hat{u}_{\eta\eta} = 0.$$

The coefficients of $\hat{u}_{\xi\xi}$ *and* $\hat{u}_{\eta\eta}$ *should be equal to zero, while the ones with* $\hat{u}_{\xi\eta}$ *shouldn't vanish. Thus taking*

$$\begin{aligned} \alpha =& 1, \quad \beta = c \\ \gamma =& -1, \quad \delta = c \end{aligned}$$

gives the desired transformation.

2. *Transforming the equation with the coefficients found above, we get*

$$-4c^2 \hat{u}_{\xi\eta}(\xi,\eta) = \hat{f}(\xi,\eta),$$

where $\hat{f}(\xi,\eta) = f(x,y)$ *and so*

$$\hat{u}_{\xi\eta} = -\frac{1}{4c^2}\hat{f}.$$

The solution to the homogeneous equation $\hat{u}_{\xi\eta} = 0$ *is* $\hat{u} = \phi(\xi) + \psi(\eta)$. *Thus the solution to the inhomogeneous equation is*

$$\hat{u}(\xi,\eta) = -\frac{1}{4c^2}\int_0^\xi \int_0^\eta \hat{f}(u,v)\,du\,dv + \phi(\xi) + \psi(\eta).$$

Next we proceed as in the script to obtain the d'Alambert formula

$$u(t,x) = \frac{1}{2}(u^0(x+ct) + u^0(x-ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} u^1(s)\,ds + \frac{1}{2c}\int_0^t \int_{x-c(t-t')}^{x+c(t-t')} f(t',x')\,dx'dt'.$$

G 3

1. *We will first prove that* $\hat{u} \in C^2(\mathbb{R} \times \mathbb{R}_+)$. *Computing the derivatives:*

$$\hat{u}_x(t,x) = \begin{cases} u_x(t,x) & \text{for } t \geq 0, x \geq 0 \\ u_x(t,-x) & \text{for } t \geq 0, x \leq 0, \end{cases}$$

$$\hat{u}_t(t,x) = \begin{cases} u_t(t,x) & \text{for } t \geq 0, x \geq 0 \\ -u_t(t,-x) & \text{for } t \geq 0, x \leq 0. \end{cases}$$

This means that the first derivatives are continuous. Now for the second derivatives:

$$\hat{u}_{xx}(t,x) = \begin{cases} u_{xx}(t,x) & \text{for } t \geq 0, x \geq 0 \\ -u_{xx}(t,-x) & \text{for } t \geq 0, x \leq 0, \end{cases}$$

$$\hat{u}_{tx}(t,x) = \begin{cases} u_{tx}(t,x) & \text{for } t \geq 0, x \geq 0 \\ u_{tx}(t,-x) & \text{for } t \geq 0, x \leq 0, \end{cases}$$

$$\hat{u}_{tt}(t,x) = \begin{cases} u_{tt}(t,x) & \text{for } t \geq 0, x \geq 0 \\ -u_{tt}(t,-x) & \text{for } t \geq 0, x \leq 0. \end{cases}$$

The mixed derivative is continuous. Taking into consideration the condition

$$u(t,0) = 0$$

and differentiating it twice with respect to time we get

$$u_{tt}(t,0) = 0.$$

This means that $\hat{u}_{tt}$ *is continuous. From the wave equation we also have*

$$u_{xx}(t,0) = u_{tt}(t,0) = 0$$

and so $\hat{u}_{xx}$ *is continuous also (the fact that* $(u^0)''(0) = 0$ *should be added to assumptions).*

The fact that $\hat{u}$ *satisfies the wave equation with the given initial conditions is not difficult to check.*

2. *The d'Alembert formula for* $\hat{u}$ *is*

$$\hat{u}(t,x) = \frac{1}{2}(\hat{u}^0(x+ct) + \hat{u}^0(x-ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} \hat{u}^1(s)ds.$$

Therefore

$$u(t,x) = \begin{cases} \frac{1}{2}(u^0(x+ct) + u^0(x-ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} u^1(s)ds & \text{for } x-ct \geq 0 \\ \frac{1}{2}(u^0(x+ct) - u^0(ct-x)) + \frac{1}{2c}\int_{ct-x}^{x+ct} u^1(s)ds & \text{for } x-ct \leq 0. \end{cases}$$