Partielle Differentialgleichungen

2. Übungen
Lösungsvorschlag

Gruppenübung

G 1 *The surface is:*

$$S(u,v) = (x(u,v), y(u,v), z(u,v)) = (u, v, 3u - 2v)$$

with $(u,v) \in \Omega = [0,1] \times [0,2]$. *Thus*

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1, \quad \frac{\partial(x,z)}{\partial(u,v)} = \begin{vmatrix} 1 & 0 \\ 3 & -2 \end{vmatrix} = -2, \quad \frac{\partial(y,z)}{\partial(u,v)} = \begin{vmatrix} 0 & 1 \\ 3 & -2 \end{vmatrix} = -3,$$

and so

$$\begin{aligned}
\iint_S f(x,y,z) dS(x,y,z) &= \iint_S (x^2 + y^2 - z) dS \\
&= \int_0^2 \int_0^1 (u^2 + v^2 - 3u + 2v)\sqrt{1 + 2^2 + 3^2} du dv \\
&= \sqrt{14} \int_0^2 \left[\frac{1}{3}u^3 + uv^2 - \frac{3}{2}u^2 + 2uv\right]_0^1 dv \\
&= \sqrt{14} \int_0^2 \left(\frac{1}{3} + v^2 - \frac{3}{2} + 2v\right) dv \\
&= \sqrt{14} \left[-\frac{7}{6}v + \frac{1}{3}v^3 + v^2\right]_0^2 \\
&= \sqrt{14} \left(-\frac{7}{3} + \frac{8}{3} + 4\right) \\
&= \frac{13}{3}\sqrt{14}.
\end{aligned}$$

G 2 *The surface parametrization is given. The determinants are*

$$\begin{aligned}
\frac{\partial(x,y)}{\partial(\phi,\theta)} &= \begin{vmatrix} -\sin\phi\sin\theta & \cos\phi\cos\theta \\ \cos\phi\sin\theta & \sin\phi\cos\theta \end{vmatrix} = -\sin\theta\cos\theta, \\
\frac{\partial(x,z)}{\partial(\phi,\theta)} &= \begin{vmatrix} -\sin\phi\sin\theta & \cos\phi\cos\theta \\ 0 & -\sin\theta \end{vmatrix} = \sin\phi\sin^2\theta, \\
\frac{\partial(y,z)}{\partial(\phi,\theta)} &= \begin{vmatrix} \cos\phi\sin\theta & \sin\phi\cos\theta \\ 0 & -\sin\theta \end{vmatrix} = -\cos\phi\sin^2\theta.
\end{aligned}$$

This gives

$$\begin{aligned}
\iint_S \vec{F}(x,y,z) d\vec{S}(x,y,z) &= \iint_\Omega \left(-\cos\phi\sin\theta \cdot \cos\phi\sin^2\theta + \sin\phi\sin\theta \cdot \sin\phi\sin^2\theta\right) d\phi d\theta \\
&= \int_0^{2\pi} \int_0^\pi \sin^3\theta\cos 2\phi d\theta d\phi = 0.
\end{aligned}$$

G 3 *We have* $\operatorname{div}\vec{F} = 0$ *and the surface* S *is a sphere of radius 1, therefore we immediately get*

$$\iint_S \vec{F} \cdot d\vec{S} = 0.$$

G 4 *Notice that*

$$\vec{u} \times \vec{v} = (u^2v^3 - v^2u^3, u^3v^1 - u^1v^3, u^1v^2 - v^1u^2)$$

and so

$$\begin{aligned}\operatorname{div}(\vec{u} \times \vec{v}) =& (u^2v^3 - v^2u^3)_x + (u^3v^1 - u^1v^3)_y + (u^1v^2 - u^2v^1)_z \\ =& u_x^2v^3 + u^2v_x^3 - v_x^2u^3 - v^2u_x^3 \\ &+ u_y^3v^1 + u^3v_y^1 - u_y^1v^3 - u^1v_y^3 \\ &+ u_z^1v^2 + u^1v_z^2 - u_z^2v^1 - u^2v_z^1.\end{aligned}$$

Now it is a matter of simple comparison of left- and right-hand sides.

Hausübung

H 1 1. *From the equations on ξ and η we compute that*

$$x = \frac{3}{25}\xi + \frac{4}{25}\eta, \quad y = \frac{3}{25}\eta - \frac{4}{25}\xi$$

and so

$$\hat{u}_\xi = u_x \cdot x_\xi + u_y \cdot y_\xi = \frac{1}{25}(3u_x - 4u_y) = 0.$$

2. *Integrating the above equation with respect to ξ gives*

$$\hat{u}(\xi, \eta) = \phi(\eta),$$

where ϕ is some continuously differentiable function ϕ. Therefore, going back to variables x and y we get

$$u(x, y) = \phi(4x + 3y).$$

3. *Transforming the equation gives*

$$25\hat{u}_\xi(\xi, \eta) = 3u_x(x, y) - 4u_y(x, y) = 25x = 3\xi + 4\eta$$

and so

$$\hat{u}_\xi = \frac{1}{25}(3\xi + 4\eta)$$

which, after integration with respect to ξ gives

$$\hat{u}(\xi, \eta) = \frac{1}{25}\left(\frac{3}{2}\xi^2 + 4\eta\xi\right) + \phi(\eta)$$

and so

$$u(x, y) = \frac{1}{25}\left(\frac{3}{2}(3x - 4y)^2 + 4(3x - 4y)(4x + 3y)\right) + \phi(4x + 3y).$$

H 2 *Take $u(x, y) = \hat{u}(\xi, \eta) = \hat{u}(ax + by, cx + dy)$. Then*

$$\begin{aligned}u_{xx} =& a^2\hat{u}_{\xi\xi} + 2ac\hat{u}_{\xi\eta} + c^2\hat{u}_{\eta\eta} \\ u_{xy} =& ab\hat{u}_{\xi\xi} + (ad + bc)\hat{u}_{\xi\eta} + cd\hat{u}_{\eta\eta} \\ u_{yy} =& b^2\hat{u}_{\xi\xi} + 2bd\hat{u}_{\xi\eta} + d^2\hat{u}_{\eta\eta}.\end{aligned}$$

Plugging this into equation ($\star\star$) we get

$$(a^2 - 2b^2 + ab)\hat{u}_{\xi\xi} + (2ac - 4bd + ad + bc)\hat{u}_{\xi\eta} + (c^2 - 2d^2 + cd)\hat{u}_{\eta\eta} = 0.$$

Thus if

$$\begin{aligned} a^2 - 2b^2 + ab =& 0 \\ c^2 - 2d^2 + cd =& 0 \\ 2ac - 4bd + ad + bc \neq& 0 \end{aligned}$$

then we get the desired tranformation. From the above it can be easily computed that

$$\xi = x + y, \quad \eta = y - 2x.$$

H 3 *Taking* $v(s) = u(x(s), y(s))$ *we have*

$$x' = 1, \quad y' = v, \quad v' = v + 1 = y' + 1$$

and so

$$v(s) = y + s + C, \quad x(s) = s + x^0$$

which gives

$$u(x, y) = x + y.$$

H 4 *The characteristic equations are*

$$t' = 1, \quad x' = v, \quad v' = 1$$

with the solution being

$$t(s) = s, \quad x(s) = \frac{1}{2}s^2 + kx^0 s + x^0, \quad v(s) = s + kx^0.$$

Solving the above for s *and* x^0 *gives*

$$s = t, \quad x^0 = \frac{x - \frac{1}{2}t^2}{kt + 1}, u(t, x) = t + k\frac{x - \frac{1}{2}t^2}{kt + 1}.$$

The denominator should not be zero: $kt \neq -1$*. But it is assumed that* $t \geq 0$*. Therefore for* $k \geq 0$ *the solution exists for all* $x \in \mathbb{R}$*,* $t \in \mathbb{R}_+$*. But for* $k < 0$ *the solution exists only for* $x \in \mathbb{R}$ *and* $t \in [0, \frac{1}{k})$*.*