Partielle Differentialgleichungen

13. Übung
Lösungsvorschlag

Gruppenübung

G 1 Lösen Sie die folgenden Anfangswertprobleme

$$\begin{aligned} u_t(x,t) &= u_{xx}(x,t) \quad \text{für } (x,t) \in (0,l) \times \mathbb{R}_+ \\ u(x,0) &= \sin\left(\frac{5\pi}{l}x\right) + x \\ u(0,t) &= 0,\ u(l,t) = l \quad \text{für } t \geq 0 \end{aligned}$$

(wobei $l > 0$) mit diesen Schritten:

1. Ziehen Sie eine Funktion ab so dass $u(0,t) = u(l,t) = 0$.

   We take
   $$v(x,t) = u(x,t) - x.$$
   Then the problem can be transformed to
   $$\begin{aligned} v_t(x,t) &= v_{xx}(x,t) \quad \textit{für } (x,t) \in (0,l) \times \mathbb{R}_+ \\ v(x,0) &= \sin\left(\frac{5\pi}{l}x\right) \\ v(0,t) &= 0,\ v(l,t) = 0 \quad \textit{für } t \geq 0 \end{aligned}$$

2. Lösen Sie 1. mit Trennung der Variablen.

   Now we solve similarly as for the Laplace equation. First, take $v(x,t) = X(x)T(t)$. Notice that the boundary conditions give $X(0) = X(l) = 0$. Moreover, the differential equation is
   $$T'X = TX''$$
   and so
   $$\frac{T'}{T} = \frac{X''}{X} = \mu.$$
   We first solve for X and get that
   $$X_k(x) = C_k \sin\left(\frac{k\pi}{l}x\right)$$
   and $\mu_k = -\frac{\pi^2 k^2}{l^2}$. The equation for T now gives
   $$T_t(t) = D_k e^{-\frac{\pi^2 k^2}{l^2}t}.$$
   Thus
   $$v_k(x,t) = A_k e^{-\frac{\pi^2 k^2}{l^2}t} \sin\left(\frac{k\pi}{l}x\right)$$
   give solutions of our equation with zero boundary data. We now have to take into consideration the initial data. Notice that v_k satisfies the initial condition if $k = 5$ and so the solution is
   $$v(x,t) = e^{-\frac{25\pi^2}{l^2}t} \sin\left(\frac{5\pi}{l}x\right)$$
   (notice that the constant is equal to 1). Finally $u(x,t) = v(x,t) + x$.

G 2 Betrachten Sie die Probleme

$$\Delta u(x,y) + u(x,y) = 0 \quad \text{für } (x,y) \in B(0,1). \tag{†}$$

1. Schreiben Sie (†) in Polarkoordinaten.

   If we introduce polar coordinates

   $$x = r\cos\phi, \quad y = r\sin\phi$$

   for $r \in [0,1)$ and $\phi \in [0,2\pi)$ then for $u(x,y) = v(r,\phi)$ we have

   $$\Delta u(x,y) = v_{rr} + \frac{1}{r}v_r + \frac{1}{r^2}v_{\phi\phi}$$

   (this is the Laplacian in polar coordinates) and (†) becomes

   $$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\phi\phi} + u = 0$$

   (we will keep writing u instead of v) and so, after multiplying by r^2 we get

   $$r^2 u_{rr} + r u_r + u_{\phi\phi} + r^2 u = 0.$$

2. Entwickeln Sie u in eine Fourierreihe.

   This is just

   $$u(r,\phi) = \sum_{n=-\infty}^{\infty} a_n(r) e^{in\phi}$$

   (we write the series with respect to ϕ and so necessarily the coefficients must depend on r).

3. Zeigen Sie, dass die Koeffiziente von diese Reihe die Gleichung

   $$r^2 a_n''(r) + r a_n'(r) + \left(r^2 - n^2\right) a_n(r) = 0$$

   erfüllen. $\{a_n\}$ heißen die *Bessel-Funktionen.*

   We differentiate the above Fourier series and get

   $$\sum_{n=-\infty}^{\infty} \left[r^2 a_n''(r) + r a_n'(r) - n^2 a_n(r) + r^2 a_n(r)\right] e^{in\phi} = 0$$

   for $\phi \in [0,2\pi)$. The above means that for fixed r we have a Fourier series which vanishes for ϕ in some interval and so these coefficients must vanish. This gives the desired equation with $n \in \mathbb{Z}$.

4. Entwickeln Sie a_n in eine Potenzreihe:

   $$a_n(r) = \sum_{k=0}^{\infty} c_k r^{k+n}$$

   und zeigen Sie, dass

   $$\sum_{k=0}^{\infty} c_k \left[(n+k)^2 - n^2\right] r^{k+n} + \sum_{k=0}^{\infty} c_k r^{k+n+2} = 0 \tag{$\circ$}$$

   für $n \geq 2$ gilt.

This is done similarly as above, we differentiate the series term by term to obtain the result.

5. Zeigen Sie, dass

$$a_n(r) = c_0 r^n \left[1 + \sum_{k=1}^{\infty} (-1)^k \frac{n!}{k!(n+k)!} \left(\frac{r}{2}\right)^{2k}\right]$$

für $n \geq 2$ erfüllt ist.

In $(\circ)$ *we have in fact a Taylor series of some function, which vanishes for* $r \in [0,1)$ *and so again all the coefficients must vanish. The trick is compare the right coefficients. The smallest power of* r *in* $(\circ)$ *is* n *and from comparing the coefficients of* r^n *we get the equation*

$$c_0[n^2 - n^2] = 0$$

and so in fact c_0 *is arbitrary. Next consider* r^{n+1}*. Again by comparing coefficients we get the equation*

$$c_1[(n+1)^2 - n^2] = 0$$

and so $c_1 = 0$*. For* $k \geq 2$*, from* $(\circ)$ *it follows that*

$$c_k\left[(n+k)^2 - n^2\right] + c_{k-2} = 0.$$

Thus, since $c_1 = 0$ *we get that also* $c_3 = 0$ *and so also* $c_5 = 0$ *and so on. Moreover*

$$c_k = -\frac{c_{k-2}}{k(2n+k)}$$

and so for $k = 2l$ *we get*

$$c_{2l} = -\frac{c_{2(l-1)}}{2l(2n+2l)} = -\frac{1}{2^2}\frac{c_{2(l-1)}}{l(n+l)}.$$

From this it is not difficult to observe by induction that

$$c_{2l} = (-1)^l \frac{1}{2^{2l}} \frac{c_0}{l!(n+1)(n+2)...(n+l)} = (-1)^l \frac{1}{2^{2l}} \frac{c_0 n!}{l!(n+l)!}$$

and this proves the desired formula.