Partielle Differentialgleichungen

12. Übung
Lösungsvorschlag

Gruppenübung

G 1 Our goal is to find *nonzero* solutions of the following equation

$$\Delta u(x,y) = \lambda u(x,y) \quad \text{for } (x,y) \in S = (0,1)^2 \tag{$\star$}$$

with the boundary condition

$$u(x,y) = 0 \quad \text{for } (x,y) \in \partial S. \tag{$\dagger$}$$

1. Assume that the solution is of the form

$$u(x,y) = X(x)Y(y),$$

where the functions X and Y are to be determined (this is called *separation of variables*). Prove that ($\star$) can be written in the form

$$\frac{X''}{X} = \lambda - \frac{Y''}{Y}$$

and that in fact both sides of the above equality are constant (from this, the functions X and Y will be computed *separately*).
Hint: Use the fact that if $g(x) = f(y)$ for two independent variables x and y then in fact $g(x) = f(y) = \mu$, where μ is a constant.

We write out the Laplacian

$$\Delta u(x,y) = X''(x)Y(y) + X(x)Y''(y).$$

Therefore ($\star$) *is equivalent to*

$$X''Y + XY'' = \lambda XY$$

and after dividing by XY (which we assume is not zero) we get the desired equality. Also, applying the hint, we have that both sides of this equality must be constant, since the functions depend on different independent variables x and y. Therefore

$$\frac{X''}{X} = \lambda - \frac{Y''}{Y} = \mu.$$

2. Using ($\dagger$) derive the conditions on $X(0)$, $X(1)$, $Y(0)$ and $Y(1)$.

Since $u(0,y) = 0$ for all y and Y we assume u at some point is not zero, then also Y at some point is not zero and therefore

$$X(0)Y(y) = 0 \quad \Leftrightarrow \quad X(0) = 0.$$

Similarly

$$X(1) = 0, \quad Y(0) = 0, \quad Y(1) = 0.$$

3. Assuming that $X'' = \mu X$ and $Y'' = (\lambda - \mu)Y$, solve first the equation for X. Using the conditions on $X(0)$ and $X(1)$ prove that only $\mu < 0$ is reasonable.

We solve $X'' = \mu X$ with the conditions $X(0) = X(1) = 0$.
Case $\mu = 0$ *We have $X'' = 0$ and so*

$$X(x) = Ax + B,$$

where A and B are some constants. But the conditions $X(0) = X(1) = 0$ imply that $A = B = 0$ and this contradicts that our solution u is nonzero.
Case $\mu > 0$ *The solution is now*

$$X(x) = Ae^{\sqrt{\mu}x} + Be^{-\sqrt{\mu}x}$$

and again the conditions $X(0) = X(1) = 0$ imply that $X \equiv 0$.
Case $\mu < 0$ *The solution is*

$$X(x) = A\cos(\sqrt{-\mu}x) + B\sin(\sqrt{-\mu}x).$$

The condition $X(0) = 0$ implies that $A = 0$. From the condition $X(1) = 0$ we get that

$$B\sin\sqrt{-\mu} = 0$$

and so we must have $\sin\sqrt{-\mu} = 0$, since if $B = 0$ then $u = 0$, which is not what we want. The equality $\sin\sqrt{-\mu} = 0$ is satisfied only if

$$\sqrt{-\mu} = \pi k,$$

where $k \in \mathbb{N}$ and so we see that the constant μ cannot be arbitrary. In fact it must be

$$\mu_k = -\pi^2 k^2$$

Therefore we get a sequence of solutions

$$X_k(x) = B_k \sin(\pi k x).$$

4. Prove that in fact $\mu = -\pi^2 k^2$, where $k \in \mathbb{N}$ and so we get solutions X_k.

   This has been proven above.

5. Solve the equation for Y and similarly, prove that $\lambda - \mu < 0$.

   The equation is now $Y'' = (\lambda - \mu_k)Y$. The reasoning is exactly the same as above, but now the constant is $\lambda - \mu_k$. The condition is now

   $$\sqrt{\mu_k - \lambda} = \pi l$$

   for $l \in \mathbb{N}$ and so

   $$\lambda_{k,l} = -\pi^2(k^2 + l^2)$$

   with solutions

   $$Y_l(y) = C_l \sin(\pi l y).$$

6. Using the conditions on $Y(0)$ and $Y(1)$, prove that $\lambda = \lambda_{k,l} = -\pi^2(k^2 + l^2)$, where $k, l \in \mathbb{N}$ and we get solutions Y_l.
   (The numbers $\lambda_{k,l}$ are called the *eigenvalues* of the operator Δ acting in S and the functions $u_{k,l}(x, y) = X_k(x)Y_l(y)$ are called its *eigenvectors*).

   This has been shown above. Notice that the equation ($\star$) possesses nonzero solutions for $\lambda = \lambda_{k,l}$ and they are of the form

   $$u_{k,l}(x, y) = A_{k,l} \sin(\pi k x)\sin(\pi l y),$$

   where $A_{k,l}$ are arbitrary constants.

Hausübung

H 1 Similarly, solve the equation ($\star$) with conditions ($\dagger$), where $S = (0, a) \times (0, b)$ with $a > 0$, $b > 0$, i.e. find the eigenvalues and eigenvectors of Δ acting in the rectangle $(0, a) \times (0, b)$.