Partielle Differentialgleichungen

10. Übung
Lösungsvorschlag

Gruppenübung

G 1 Sei $\{u_n\}$ eine Folge harmonischer Funktionen, die in U definiert sind, und die gleichmäßig auf jeder kompakten Teilmenge von U gegen u konvergiert sind (kompakte Konvergenz). Zeigen Sie, dass u harmonisch in U ist.
Hinweis: Nutzen Sie die (umgekehrte) Mittelwertformel.

Take $B_r(x) \subset \overline{B}_r(x) \subset U$. Then the middle-value formula holds for u_n:

$$u_n(x) = \frac{n}{\omega_n r^n} \int_{B_r(x)} u_n(y)dy.$$

From the assumption we know that u_n converges uniformly to u on the set $\overline{B}_r(x)$, which means that

$$u(x) = \frac{n}{\omega_n r^n} \int_{B_r(x)} u(y)dy.$$

This holds for any ball inside of U and thus from the inverse middle-value theorem we get that u is harmonic in U.

G 2 1. Es sei

$$f(z) = \sum_{n=0}^{\infty} c_n z^n \tag{†}$$

eine in $\{z \in \mathbb{C}\colon |z| \leq 1\}$ konvergente Potenzreihe mit $c_n = a_n + ib_n \in \mathbb{C}$. Zeigen Sie, dass für den Realteil $\Re f(z)$ gilt

$$\Re f(z) = \sum_{n=0}^{\infty} r^n(a_n \cos(n\phi) - b_n \sin(n\phi)), \quad 0 \leq \phi < 2\pi, \quad 0 \leq r \leq 1$$

mit $\phi = \arg z$, $r = |z|$ (also $z = re^{i\phi}$).

This is not a difficult, straightforward calculation.

2. Lösen Sie

$$\begin{aligned} \Delta u(x,y) &= 0 \quad \text{for } x^2 + y^2 < 1 \\ u(x,y) &= 2x^2 + x - 1 \quad \text{for } x^2 + y^2 = 1. \end{aligned}$$

Since u is harmonic, it can be written as a real part of some holomorphic function: $u = \Re f$. Moreover, since u is harmonic in the circle $|z| < 1$, the power series for f is convergent in this circle. Thus

$$u(x,y) = \sum_{n=0}^{\infty} r^n(a_n \cos(n\phi) - b_n \sin(n\phi))$$

for $0 \leq \phi < 2\pi$, $0 \leq r < 1$ and $x = r\cos\phi$, $y = r\sin\phi$. But for $r = 1$ we have the boundary value:

$$u(x,y) = 2x^2 + x - 1 = 2\cos^2\phi + \cos\phi - 1.$$

Using the formula $\cos^2\phi = \frac{1}{2}(\cos 2\phi + 1)$ we get

$$u(x,y) = \cos(2\phi) + \cos\phi.$$

Therefore, comparing the coefficients we obtain

$$a_1 = 1, a_2 = 1, a_n = 0 \text{ for other } n, b_n = 0 \text{ for all } n.$$

This means that for $0 \leq r \leq 1$ *we have*

$$u(x, y) = r \cos \phi + r^2 \cos(2\phi).$$

But if $x = r \cos \phi$ *and* $y = r \sin \phi$ *then*

$$r^2 \cos(2\phi) = r^2 \cos^2 \phi - r^2 \sin^2 \phi = x^2 - y^2$$

and so

$$u(x, y) = x^2 - y^2 + x.$$