Partielle Differentialgleichungen

1. Übungen
Lösungsvorschlag

Gruppenübung

G 1 The Burgers' equation

1.

$$v'(s) = u_x(t(s), x(s))t'(s) + u_y(t(s), x(s))x'(s).$$

2. The conditions:

$$t'(s) = 1, \quad x'(s) = u(t(s), x(s)) = v(s).$$

If $v(s) = K$ then

$$t(s) = s + c_1, \quad x(s) = Ks + c_2.$$

The curve should start at the line OX, since we have the initial condition there (i.e. solution is known there). This gives:

$$t(s) = s, \quad x(s) = Ks + x^0.$$

But if we plug in $s = 0$ then from the initial condition

$$K = v(0) = u(0, x^0) = u^0(x^0).$$

Therefore $t(s) = s, x(s) = u^0(x^0)s + x^0$.

3. (a) $t(s) = s, x(s) = s + x^0$.

(b) $t(s) = s, x(s) = -s + x^0$.

(c)

$$t(s) = s, x(s) = \begin{cases} s + x^0 & \text{for } x^0 < 0 \\ (1 - x^0)s + x^0 & \text{for } 0 \le x^0 < 1 \\ x^0 & \text{for } 1 \le x^0 \end{cases}$$

(d) All of the above are lines, starting at point $(x^0, 0)$. In (c), the lines which start for $0 \le x^0 \le 1$ all meet at point $(1, 1)$. This means that at this point the solution would have to take all values from the interval $[0, 1]$, and so it doesn't exist at $(1, 1)$.

Hausübung

H 1 *In both exercises the solution along the curves is no more constant. We again take $v(s) = u(t(s), x(s))$ and compare the derivative of v to the equation. This gives*

1.

$$t'(s) = 1, \; x'(s) = \alpha, \; v'(s) = 1$$

The solutions are

$$t(s) = s, \; x(s) = \alpha s + x^0, \; v(s) = s + u^0(x^0)$$

If we want to find solution at given point, i.e. $u(\hat{t}, \hat{x})$ then we have solve the equations to find s and x^0:

$$\hat{t} = s, \; \hat{x} = \alpha s + x^0$$

which gives

$$s = \hat{t}, \; x^0 = \hat{x} - \alpha\hat{t}$$

and thus $u(\hat{t}, \hat{x}) = v(s) = \hat{t} + u^0(\hat{x} - \alpha\hat{t})$. This gives an explicit formula for u.

2.

$$t'(s) = 1,\ x'(s) = \alpha,\ v'(s) = \beta u(t(s), x(s)) = \beta v(s)$$

Thus the solutions are

$$t(s) = s,\ x(s) = \alpha s + x^0,\ v(s) = u^0(x^0)e^{\beta s}$$

Again, if we want to find an explicit formula for $u(\hat{t}, \hat{x})$ *we solve the equations*

$$\hat{t} = s,\ \hat{x} = \alpha s + x^0$$

which gives

$$s = \hat{t},\ x^0 = \hat{x} - \alpha s$$

and therefore $u(\hat{t}, \hat{x}) = u^0(\hat{x} - \alpha s)e^{\beta\hat{t}}$.