Fachbereich Mathematik
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Partielle Differentialgleichungen

7. Übung

Gruppenübung

G 1 Let $u \in C^3(\Omega)$ be a harmonic function in an open set $\Omega \in \mathbb{R}^n$, i.e.

$$\Delta u(x) = 0 \quad \text{for } x \in \Omega.$$

1. Prove that $w(x) = x \cdot \nabla u(x)$ is also harmonic in Ω.
2. Prove that
$$\Delta \left(u^2(x)\right) \geq 0, \quad \Delta \left(e^{u(x)}\right) \geq 0.$$

G 2 Let $f\colon D \to \mathbb{C}$ be a holomorphic function, with $D \subset \mathbb{C}$ and let $f(x, y) = u(x, y) + iv(x, y)$. Then u and v satisfy the *Cauchy-Riemann equations*:

$$u_x = v_y, \quad u_y = -v_x.$$

1. Conclude that the functions u and v are harmonic in D.
2. Find v if $u(x, y) = xy^3 - x^3 y$.
3. Prove that v is a polynomial if u is a polynomial.

Hausübung

H 1 Let u be harmonic in $\mathbb{R}^n$, $x_0 \in \mathbb{R}^n$ and A be an $n \times n$ orthonormal matrix, i.e. $A \cdot A^T = A^T \cdot A = I$. Prove that then $v(x) = u(A(x - x_0))$ is also harmonic.

H 2 Let $u \in C^2(\Omega)$, with $\Omega \subset \mathbb{R}^n$ open and bounded, satisfy the following von Neumann problem

$$\begin{aligned} -\Delta u(x) =& f(x) \quad \text{for } x \in \Omega \\ \frac{\partial u}{\partial \nu}(x) =& h(x) \quad \text{for } x \in \partial\Omega \end{aligned}$$

where $\frac{\partial u}{\partial \nu} = \nabla u \cdot \nu$ is the normal derivative. Prove the *compatibility condition*

$$\int_\Omega f(x)dx + \int_{\partial\Omega} h(x)dS_x = 0.$$