Fachbereich Mathematik
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Partielle Differentialgleichungen

6. Übung

Gruppenübung

G 1 The Euler equations of gas dynamics are

$$\begin{aligned} \rho \partial_t u + k\rho^\alpha \nabla \rho &= 0 \\ \partial_t \rho + \operatorname{div}(\rho u) &= 0 \end{aligned}$$

with constants $k, \alpha > 0$ and with the velocity $u\colon \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$ and the density $\rho\colon \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}_+$.

1. Assume that $|u(t,x)|$ is small and that the derivatives $|\nabla u(t,x)|$, $|\partial_t u(t,x)|$ are small. Assume also that $\rho(t,x) = \rho_0 + \rho_1(t,x)$, where $\rho_0 > 0$ is a constant and where $|\rho_1(t,x)|$ is small with small derivatives. Show that approximately the equations

$$\rho_0 \partial_t u + k\rho_0^\alpha \nabla \rho_1 = 0 \qquad (1)$$
$$\partial_t \rho_1 + \rho_0 \operatorname{div} u = 0 \qquad (2)$$

hold.
Hint: Drop all expressions, which contain products of small terms and use

$$(\rho_0 + \rho_1)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} \rho_0^{\alpha-k} \rho_1^k, \quad \left|\frac{\rho_1}{\rho_0}\right| < 1.$$

2. If (1) and (2) hold, prove that ρ_1 satisfies the wave equation

$$\partial_t^2 \rho_1 = c^2 \Delta \rho_1$$

with $c^2 = k\rho_0^\alpha$.

G 2 Using Kirchoff's formula compute the values $u(t,0)$ for the solution of the initial value problem

$$\begin{aligned} u_{tt} &= c^2 \Delta u \quad \text{for } (t,x) \in \mathbb{R}_+ \times \mathbb{R}^3 \\ u(0,x) &= \frac{1}{1+|x|^2} \\ u_t(0,x) &= 0. \end{aligned}$$

G 3 Let u be a C^2 solution of the inhomogeneous wave equation

$$u_{tt}(t,x) = \Delta u(t,x) + f(t,x)$$

for $(t,x) \in [0,T] \times \Omega$, with $\Omega \subset \mathbb{R}^n$ being an open and bounded set. Let u satisfy the *Neumann boundary condition*

$$\frac{\partial u}{\partial n}(t,x) = h(t,x) \quad \text{for } (t,x) \in [0,T] \times \partial\Omega,$$

where $\frac{\partial u}{\partial n}(t,x) = \nabla u(t,x) \cdot n(x)$ is the normal derivative of u.

1. Prove that if $\int_\Omega u_t(t^\star, x)dx = 0$ for some $t^\star \in [0,T]$ and
$$\int_{\partial\Omega} h(t,x)dS_x = 0, \quad \int_\Omega f(t,x)dx = 0$$
for all $t \in [0,T]$, then
$$\int_\Omega u_t(t,x)dx = 0$$
for all $t \in [0,T]$.
2. Prove that if additionally $\int_\Omega u(t^{\star\star}, x)dx = 0$ for some $t^{\star\star} \in [0,T]$, then
$$\int_\Omega u(t,x)dx = 0$$
for all $t \in [0,T]$.

Hausübung

H 1 Let $u(t, x_1, x_2)$ be a solution of the two-dimensional wave equation

$$\begin{aligned} \partial_t^2 u =& c^2 \Delta_2 u, \quad \text{for } (t,x) \in \mathbb{R}_+ \times \mathbb{R}^2 \\ u(0,x) =& 0 \\ \partial_t u(0,x) =& u_1(x). \end{aligned}$$

Let $v(t, x_1, x_2, x_3)$ be the solution of the three-dimensional wave equation

$$\begin{aligned} \partial_t^2 v =& c^2 \Delta_3 v, \quad \text{for } (t,x) \in \mathbb{R}_+ \times \mathbb{R}^3 \\ v(0, x_1, x_2, x_3) =& 0 \\ \partial_t v(0, x_1, x_2, x_3) =& u_1(x_1, x_2) x_3. \end{aligned}$$

Prove that
$$v(t, x_1, x_2, x_3) = u(t, x_1, x_2) x_3.$$

Hint: Prove that
$$v(t,x) = \frac{1}{2\pi c} \int_{|y'-x'|<ct} \frac{u_1(y') x_3}{\sqrt{c^2t^2 - |x'-y'|^2}} dy',$$
where $x = (x_1, x_2, x_3)$, $x' = (x_1, x_2)$, $y' = (y_1, y_2)$.