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Partielle Differentialgleichungen

5. Übung

Hausübung

H 1 Let u be the solution to the initial-boundary problem

$$\begin{aligned} u_{tt} =& c^2\Delta u + f \quad \text{for } (t,x) \in \mathbb{R}_+ \times \Omega \\ u(t,x) =& 0 \quad \text{for } (t,x) \in \mathbb{R}_+ \times \partial\Omega \\ u(0,x) =& u^0(x) \quad \text{for } x \in \Omega \\ u_t(0,x) =& u^1(x) \quad \text{for } x \in \Omega, \end{aligned}$$

where $\Omega \subset \mathbb{R}^n$ is open and bounded. Let

$$E(t) = \int_\Omega \left(u_t^2 + c^2|\nabla u|^2\right) dx$$

be the energy at point $t \in \mathbb{R}_+$.

1. Prove that if $f = 0$ then $E(t)$ is a constant function.
   Hint: Compute the derivative and use integration by parts.
2. Prove that the solution with arbitrary f is unique.
   Hint: Assume there are two solutions u and v. What initial-boundary problem does the difference $u - v$ satisfy?

H 2 Let $u \in C^2(\mathbb{R}^3)$ be a solution of the partial differential equation

$$-\Delta u(x) = g(x), \quad x \in \mathbb{R}^3.$$

1. Prove that
$$\frac{1}{r^2}\int_{B_r(x)} \Delta u(y)dy = \frac{d}{dr}\left(\frac{1}{r^2}\int_{\partial B_r(x)} u(y)dS_y\right).$$
2. Use this equation to show that
$$u(x) = \frac{1}{\omega_3 r^2}\int_{\partial B_r(x)} u(y)dS_y + \int_0^r \left(\frac{1}{\omega_3 r^2}\int_{B_r(x)} g(y)dy\right) dr.$$