Fachbereich Mathematik
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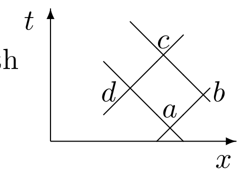
Partielle Differentialgleichungen

4. Übung

Gruppenübung

G 1 Suppose that $u \in C^2(\mathbb{R} \times \mathbb{R}_+)$ solves the wave equation:

$$u_{tt}(x,t) - u_{xx}(x,t) = 0 \quad \text{for } (t,x) \in \mathbb{R} \times \mathbb{R}_+.$$



Let $a, b, c, d \in \mathbb{R} \times \mathbb{R}_+$ be the consecutive vertices of a rectangle with sides parallel to the lines $x = t$ and $x = -t$.
Recall that the solution is of the form $u(x,t) = \phi(x+t) + \psi(x-t)$.

1. Prove that $u(a) + u(c) = u(b) + u(d)$.
2. Prove that there exists at most one solution to the following initial-boundary problem

$$\begin{aligned} u_{tt} =& u_{xx} \quad \text{for } (x,t) \in (0,l) \times \mathbb{R}_+ \\ u(0,t) =& \alpha(t), \quad u(l,t) = \beta(t) \\ u(x,0) =& g(x), \quad u_t(x,0) = h(x). \end{aligned}$$

   What conditions should the functions α, β, g, h satisfy for the solution to be of class $C^2([0,l] \times [0,\infty))$?
3. **Goursat's problem**
   Find the solution to the wave equation $u_{tt} = u_{xx}$ in $\mathbb{R} \times \mathbb{R}$ such that $u(t+1,t) = \alpha(t)$ and $u(-t+5,t) = \beta(t)$. What conditions should the functions α, β satisfy for this solution to be of class $C^2(\mathbb{R}^2)$?

Hausübung

H 1 Solve the following initial-boundary problem

$$\begin{aligned} u_{tt} =& u_{xx} \quad \text{for } (x,t) \in \mathbb{R}^2_+ \\ u(0,x) =& x^2 \\ u_t(0,x) =& x \\ u(t,0) =& t^2. \end{aligned}$$

Hint: Take $w(t,x) = u(t,x) - t^2$ and solve a corresponding inhomogeneous problem for w, using the reflection principle from previous exercises.

H 2 Suppose $u \in C^3(\mathbb{R}^n \times \mathbb{R}_+)$ solves the n-dimensional wave equation:

$$u_{tt} = \Delta u.$$

Prove that then

$$v(x,t) = x \cdot \nabla u(x,t) + t u_t(x,t)$$

is also a solution to the wave equation ($\cdot$ is the scalar product).