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Partielle Differentialgleichungen

3. Übung

Gruppenübung

G 1 Suppose that u solves the wave equation:

$$u_{tt} - u_{xx} = 0.$$

Prove that

1. $v = u_x + u_t$
2. $v = u_x^2 + u_t^2$

also solve the wave equation.

G 2 Suppose that u satisfies the wave equation

$$u_{tt} = c^2 u_{xx}.$$

1. Find a linear change of variables

$$\xi = \alpha x + \beta t, \quad \eta = \gamma x + \delta t$$

such that the function $\hat{u}(\xi, \eta) = u(t, x)$ satisfies the equation

$$\hat{u}_{\xi\eta} = 0.$$

2. Solve then the inhomogeneous problem

$$\begin{aligned} u_{tt} - c^2 u_{xx} &= f \\ u(0, x) &= u^0(x) \\ u_t(0, x) &= u^1(x). \end{aligned}$$

G 3 Suppose that $u \in C^2(\mathbb{R}_+ \times \mathbb{R}_+)$ satisfies the wave equation on the half-line:

$$\begin{aligned} u_{tt}(t, x) &= u_{xx}(t, x) \quad \text{for } (t, x) \in \mathbb{R}_+ \times \mathbb{R}_+ \\ u(0, x) &= u^0(x) \quad \text{for } x \in \mathbb{R}_+ \\ u_t(0, x) &= u^1(x) \quad \text{for } x \in \mathbb{R}_+ \\ u(t, 0) &= 0 \end{aligned} \qquad (\star)$$

with $u^0(0) = 0$, $u^1(0) = 0$. Define

$$\hat{u}(t, x) = \begin{cases} u(t, x) & \text{for } t \geq 0,\ x \geq 0 \\ -u(t, -x) & \text{for } t \geq 0,\ x \leq 0, \end{cases}$$

$$\hat{u}^0(x) = \begin{cases} u^0(x) & \text{for } x \geq 0 \\ -u^0(-x) & \text{for } x \leq 0, \end{cases}$$

$$\hat{u}^1(x) = \begin{cases} u^1(x) & \text{for } x \geq 0 \\ -u^1(-x) & \text{for } x \leq 0. \end{cases}$$

1. Prove that $\hat{u} \in C^2(\mathbb{R}_+ \times \mathbb{R})$ and $\hat{u}$ solves the problem

$$\begin{aligned} \hat{u}_{tt} =& \hat{u}_{xx} \quad \text{for}(t,x) \in \mathbb{R}_+ \times \mathbb{R} \\ \hat{u}(0,x) =& \hat{u}^0(x) \\ \hat{u}_t(0,x) =& \hat{u}^1(x). \end{aligned} \tag{$\star\star$}$$

2. Write the d'Alambert equations for $(\star\star)$ and then modify them to find an analogous formula for solving $(\star)$.

Hausübung

H 1 Suppose that $g, h \in C^\infty(\mathbb{R}_+)$ and let

$$u(t,x) = \sum_{k=0}^{\infty} \left[\frac{t^{2k}}{(2k)!} \Delta^{(k)} g(x) + \frac{t^{2k+1}}{(2k+1)!} \Delta^{(k)} h(x) \right], \tag{$\dagger$}$$

where $\Delta^{(0)} f(x) = f(x)$, $\Delta f(x) = \Delta^{(1)} f(x) = f_{xx}(x)$ and $\Delta^{(k+1)} f(x) = \Delta^{(k)} \Delta f(x)$. Suppose that the series $(\dagger)$ is absolutely convergent for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$ and it can be differentiated two times, term by term, with respect to t and x.

1. Prove that u solves the wave equation $u_{tt} = u_{xx}$ with the initial conditions $u(0,x) = g(x)$ and $u_t(0,x) = h(x)$.
2. Solve the wave equation with initial data:
   (a) $g(x) = x^2$, $h(x) = 1$
   (b) $\Delta g(x) = \Delta h(x) = 0$
   (c) $g(x) = e^x$, $h(x) = e^{-x}$.