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Partielle Differentialgleichungen

2. Übung

Gruppenübung

G 1 Suppose that a surface S has parametrization

$$S\colon \begin{cases} x = x(u,v) \\ y = y(u,v) \\ z = z(u,v) \end{cases}$$

for $(u,v) \in \Omega \subset \mathbb{R}^2$. Let $f\colon S \to \mathbb{R}$ be a function. Let

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}, \quad \frac{\partial(x,z)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ z_u & z_v \end{vmatrix}, \quad \frac{\partial(y,z)}{\partial(u,v)} = \begin{vmatrix} y_u & y_v \\ z_u & z_v \end{vmatrix}.$$

Then the *surface integral of first kind* is defined by

$$\iint_S f(x,y,z)dS(x,y,z) =$$
$$\iint_\Omega f(x(u,v),y(u,v),z(u,v))\sqrt{\left[\frac{\partial(x,y)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(x,z)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(y,z)}{\partial(u,v)}\right]^2}\,dudv.$$

Let $S = \{(x,y,z) \in \mathbb{R}^3 \colon z = 3x - 2y\}$ and let $f(x,y,z) = x^2 + y^2 - z$. Compute $\iint_S f dS$.

G 2 Suppose S has parametrization as in **G 1**. Let $\vec{F} = [P,Q,R]\colon S \to \mathbb{R}^3$ be a vector field. Then the *surface integral of the second kind* is defined by

$$\begin{aligned}\iint_S \vec{F}(x,y,z)d\vec{S} = &\iint_\Omega P(x(u,v),y(u,v),z(u,v))\frac{\partial(y,z)}{\partial(u,v)}dudv- \\ &-\iint_\Omega Q(x(u,v),y(u,v),z(u,v))\frac{\partial(x,z)}{\partial(u,v)}dudv+ \\ &+\iint_\Omega R(x(u,v),y(u,v),z(u,v))\frac{\partial(x,y)}{\partial(u,v)}dudv\end{aligned}$$

or, in shorter form

$$\iint_S \vec{F}(x,y,z)d\vec{S}(x,y,z) = \iint_\Omega \vec{F} \circ \vec{S}(u,v) \cdot \left(\frac{\partial \vec{S}}{\partial u} \times \frac{\partial \vec{S}}{\partial v}\right) dudv,$$

where $\vec{S}(u,v) = (x(u,v), y(u,v), z(u,v))$. The vector $\vec{n}(u,v) = \frac{\partial \vec{S}}{\partial u} \times \frac{\partial \vec{S}}{\partial v}$ is called the *normal vector* to the surface S at point (u,v).

Let $\vec{S} = (\cos\phi \sin\theta, \sin\phi \sin\theta, \cos\theta)$ for $(\phi,\theta) \in [0,2\pi] \times [0,\pi]$ and let $\vec{F}(x,y,z) = [x,-y,0]$. Sketch the surface S. Compute $\iint_S \vec{F}d\vec{S}$.

G 3 **Gaußscher Integralsatz** Let $V \subset \mathbb{R}^3$ be an open compact set, with boundary surface ∂V possessing an outward normal vector $\vec{n}$ at each point and let $\vec{F} = [P, Q, R] \colon V \to \mathbb{R}^3$ be a C^1 vector field. Then

$$\iiint_V \operatorname{div} \vec{F} dV = \iint_{\partial V} \vec{F} d\vec{S},$$

where $\operatorname{div} \vec{F} = P_x + Q_y + R_z$.

Use Gauss's theorem to compute the integral from **G 2**.

G 4 Let $\vec{u}, \vec{v} \colon \mathbb{R}^3 \to \mathbb{R}^3$ be of class C^1, let $\vec{u} = (u^1, u^2, u^3)$, $\vec{v} = (v^1, v^2, v^3)$ and put

$$\operatorname{rot} \vec{u} = (u^3_y - u^2_z, u^1_z - u^3_x, u^2_x - u^1_y).$$

Prove that

$$\operatorname{div}(\vec{u} \times \vec{v}) = \vec{v} \cdot \operatorname{rot} \vec{u} - \vec{u} \cdot \operatorname{rot} \vec{v}.$$

For homework only 2 exercises are required: $(H1 \vee H2) \wedge (H3 \vee H4)$**.**

Hausübung

H 1 Consider the equation

$$3u_x(x, y) - 4u_y(x, y) = 0 \quad \text{in } \mathbb{R}^2. \tag{$\star$}$$

1. Prove that if we take

$$\xi = 3x - 4y, \quad \eta = 4x + 3y$$

and $\hat{u}(\xi, \eta) = u(x(\xi, \eta), y(\xi, \eta))$ then the equation $(\star)$ takes the form $\hat{u}_\xi = 0$.

2. Solve $(\star)$.
3. In a similar way solve the equation

$$3u_x(x, y) - 4u_y(x, y) = 25x.$$

H 2 Find a linear transformation of variables

$$\xi = ax + by, \quad \eta = cx + dy,$$

which transforms the equation

$$u_{xx}(x, y) + u_{xy}(x, y) - 2u_{yy}(x, y) = 0 \quad \text{for } (x, y) \in \mathbb{R}^2 \tag{$\star\star$}$$

into the equation $\hat{u}_{\xi\eta} = 0$ and then solve $(\star\star)$.

H 3 Proceeding similarly as in the case of Burgers' equation, solve

$$u_x(x, y) + u(x, y)u_y(x, y) = u(x, y) + 1 \quad \text{for } (x, y) \in \mathbb{R}^2,$$

with the boundary condition $u(x, x) = 2x$.

H 4 Consider the initial problem

$$\begin{aligned} u_t(t, x) + u(t, x)u_x(t, x) &= 1 \quad \text{for} (t, x) \in \mathbb{R}_+ \times \mathbb{R} \\ u(0, x) &= kx \end{aligned}$$

with some constant $k \in \mathbb{R}$. Find the solutions to this problem. For what values of k, t, x do these solutions exist?