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Partielle Differentialgleichungen

12. Übung

Gruppenübung

G 1 Our goal is to find *nonzero* solutions of the following equation

$$\Delta u(x,y) = \lambda u(x,y) \quad \text{for } (x,y) \in S = (0,1)^2 \tag{$\star$}$$

with the boundary condition

$$u(x,y) = 0 \quad \text{for } (x,y) \in \partial S. \tag{$\dagger$}$$

1. Assume that the solution is of the form

   $$u(x,y) = X(x)Y(y),$$

   where the functions X and Y are to be determined (this is called *separation of variables*). Prove that ($\star$) can be written in the form

   $$\frac{X''}{X} = \lambda - \frac{Y''}{Y}$$

   and that in fact both sides of the above equality are constant (from this, the functions X and Y will be computed *separately*).
   Hint: Use the fact that if $g(x) = f(y)$ for two independent variables x and y then in fact $g(x) = f(y) = \mu$, where μ is a constant.
2. Using ($\dagger$) derive the conditions on $X(0)$, $X(1)$, $Y(0)$ and $Y(1)$.
3. Assuming that $X'' = \mu X$ and $Y'' = (\lambda - \mu)Y$, solve first the equation for X. Using the conditions on $X(0)$ and $X(1)$ prove that only $\mu < 0$ is reasonable.
4. Prove that in fact $\mu = -\pi^2 k^2$, where $k \in \mathbb{N}$ and so we get solutions X_k.
5. Solve the equation for Y and similarly, prove that $\lambda - \mu < 0$.
6. Using the conditions on $Y(0)$ and $Y(1)$, prove that $\lambda = \lambda_{k,l} = -\pi^2(k^2 + l^2)$, where $k, l \in \mathbb{N}$ and we get solutions Y_l.
   (The numbers $\lambda_{k,l}$ are called the *eigenvalues* of the operator Δ acting in S and the functions $u_{k,l}(x,y) = X_k(x)Y_l(y)$ are called its *eigenvectors*).

Hausübung

H 1 Similarly, solve the equation ($\star$) with conditions ($\dagger$), where $S = (0,a) \times (0,b)$ with $a > 0$, $b > 0$, i.e. find the eigenvalues and eigenvectors of Δ acting in the rectangle $(0,a) \times (0,b)$.