9, $f: [a,b] \to \mathbb{R}$ stetig

$f': \,]a,b[\, \to \mathbb{R}$

14.6 Mittelwertsatz

$$\exists\, p \in [a,b] \quad f'(p) = \frac{f(b) - f(a)}{b-a}$$

Eindeutigkeitssatz

$$f'(x) = g'(x) \quad \forall x \in \,]a,b[$$

$$\Rightarrow \exists\, c \in \mathbb{R} \quad g(x) = f(x) + c \quad \forall x \in [a,b]$$

14.5 $p \in \,]a,b[\,$ $f(p) = $ Max | Min

$$\rightarrow f'(p) = 0$$

14.7

$f' \geq 0 \Rightarrow f$ monoton wachsend

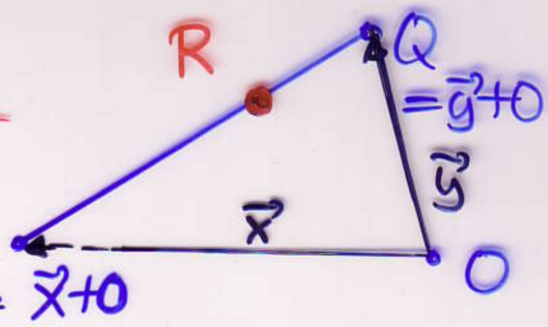
$f' > 0 \Rightarrow f$ streng mon. wachsend

$f' \leq 0 \Rightarrow f$ monoton fallend

$f' < 0 \rightarrow f$ streng mon. fallend

14.19

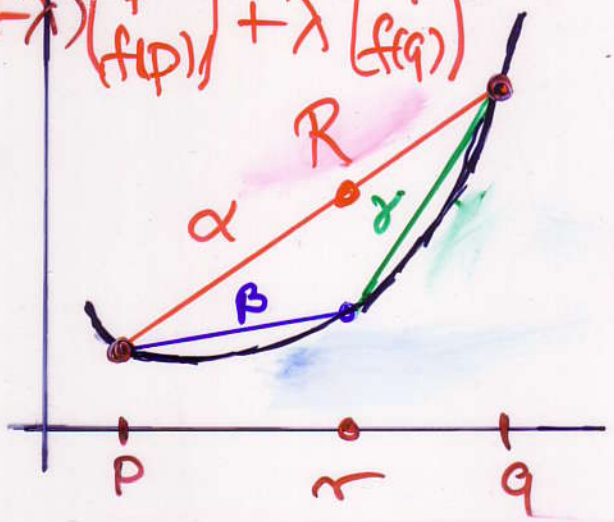
Verbindungsstrecke



$$R = \vec{x} + \lambda(\vec{y} - \vec{x}) + O = (1-\lambda)\vec{x} + \lambda\vec{y} + O$$

$$\lambda \in [0,1]$$

$$R \leq (1-\lambda)\begin{pmatrix} p \\ f(p) \end{pmatrix} + \lambda \begin{pmatrix} q \\ f(q) \end{pmatrix}$$



f konvex:

$$f(r) \leq (1-\lambda)f(p) + \lambda f(q)$$

für

$$r = (1-\lambda)p + \lambda q$$

alle $p < r < q$

Satz: f konvex

$\Leftrightarrow$ f' monoton wachsend

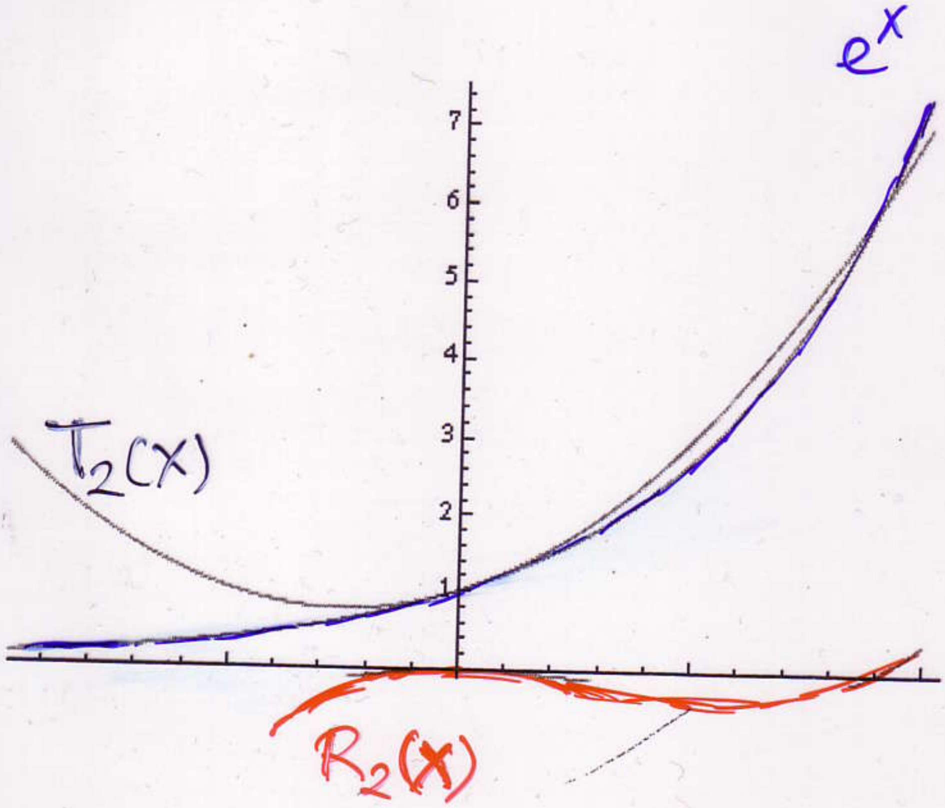
($\Leftrightarrow$ $f'' \geq 0$ falls $f'' :]a,b[\to \mathbb{R}$)

Bew. Für Sekantensteigungen

$$\beta \leq \alpha \leq \gamma$$

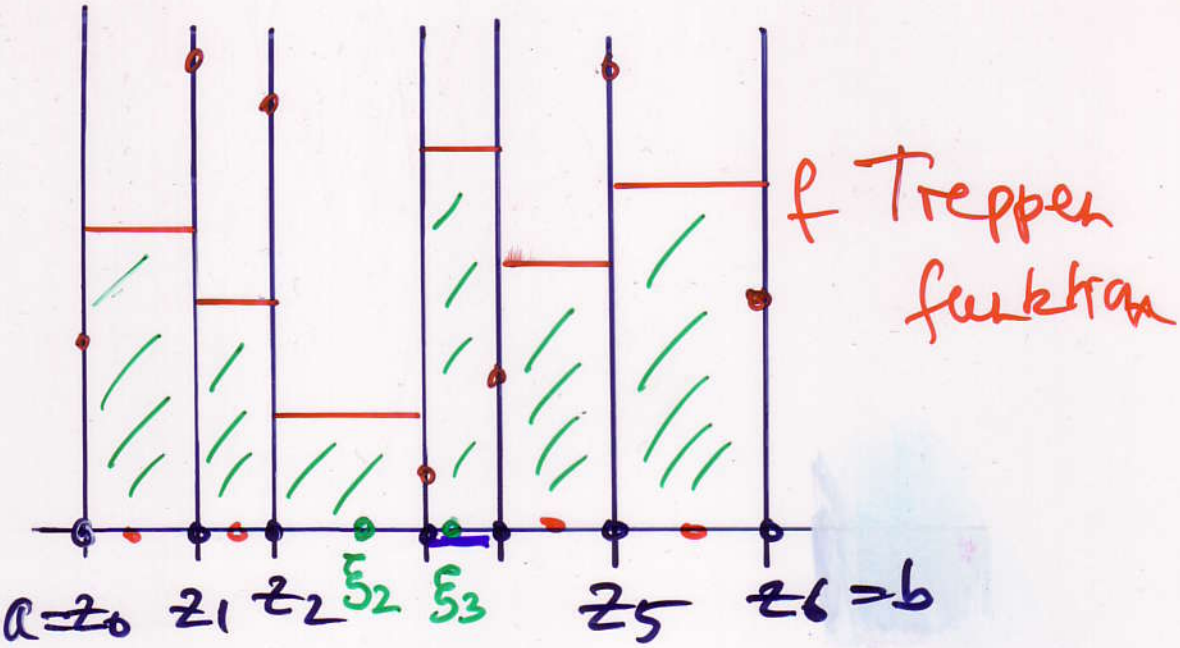
Beisp $\exp$, $\cos : [\pi/2, 3\pi/2] \to \mathbb{R}$

konkav: $\geq$ statt $\leq$



$a = p = 0$

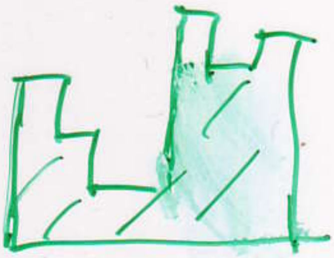
15.1.1. $f:[a,b]\to\mathbb{R}$ beschränkt



Zerlegung $(\Delta x)_k = z_{k+1} - z_k$

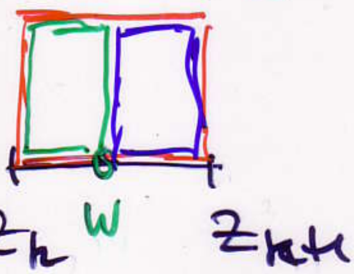
Linienelement

Maschenweite $\max_k (\Delta x)_k$

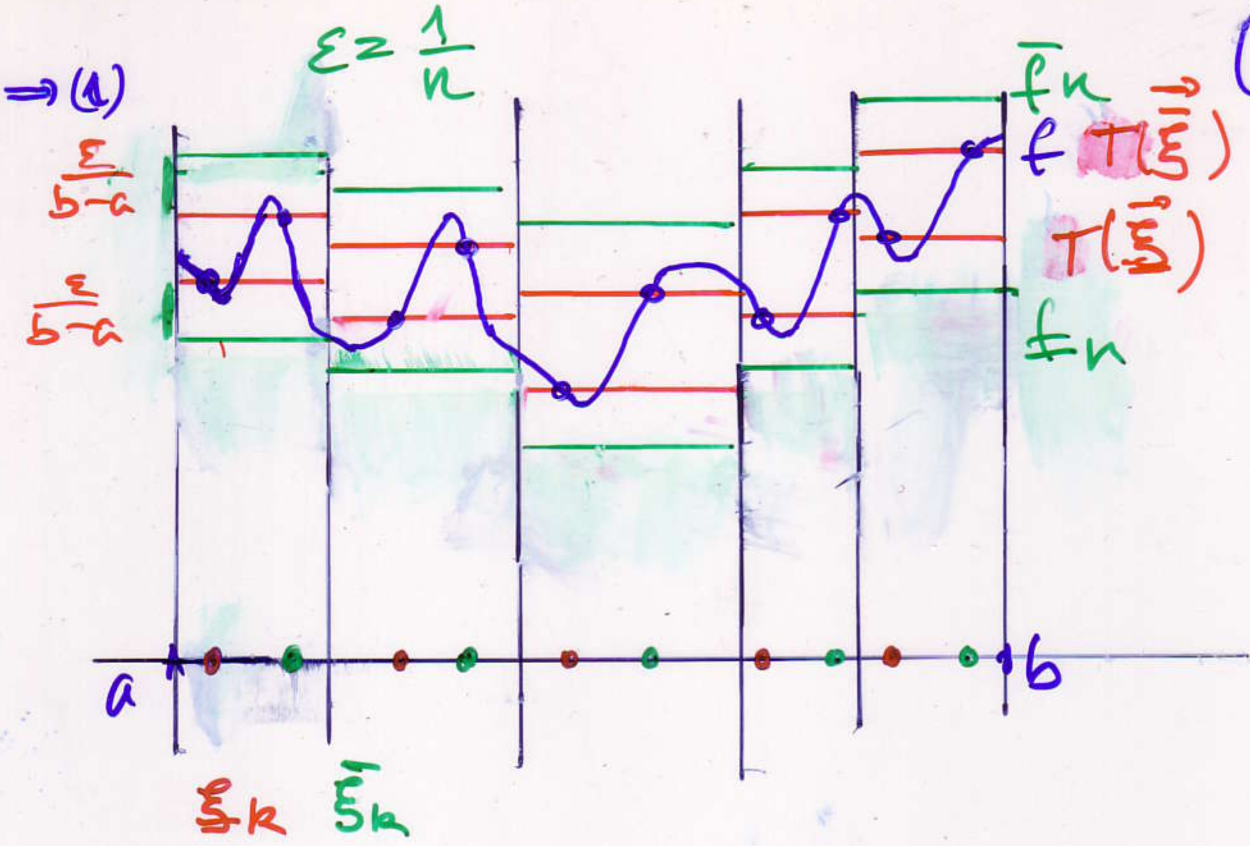
 $= \int_a^b f = \sum_{k=0}^{n-1} f(\xi_k)(\Delta x)_k$

Lemma: unabh. von Zerlegung

Bew Verfeinerung



(III) $\Rightarrow$ (1)



$$\int T(\vec{\underline{\xi}}) - \int \underline{f}_n \le \varepsilon$$

$$|c - R(\vec{\underline{\xi}})| \le \varepsilon$$

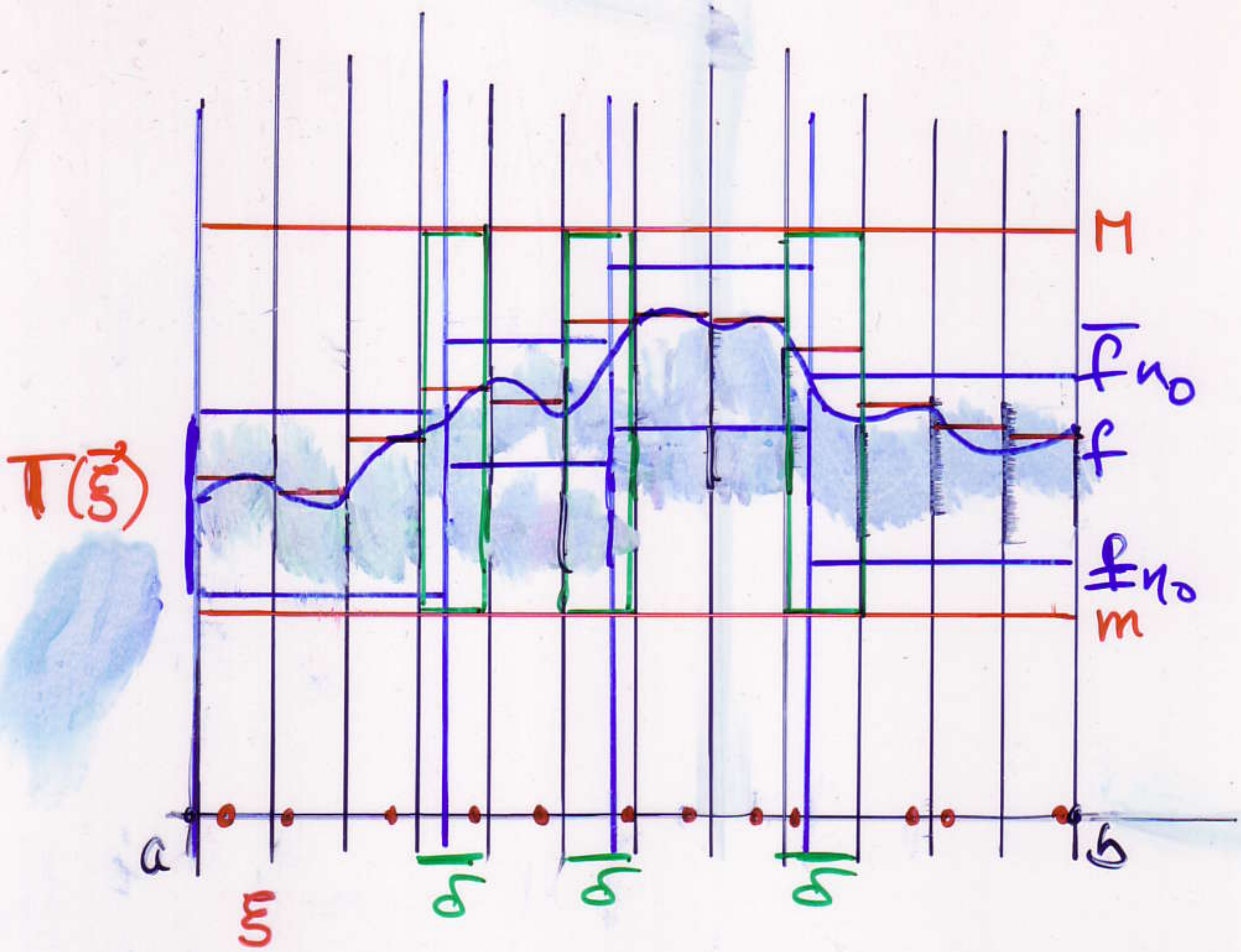
$$\left|\int \underline{f}_n - c\right| \le 2\varepsilon$$

$$\int \underline{f}_n \to c$$

$$T(\vec{\xi})(x) = \begin{cases} f(\xi_k) & z_k < x < z_{k+1} \\ f(z_k) & x = z_k \end{cases}$$

$$R(\vec{\xi}) = \int T(\vec{\xi})$$

(i) $\Rightarrow$ (ii)



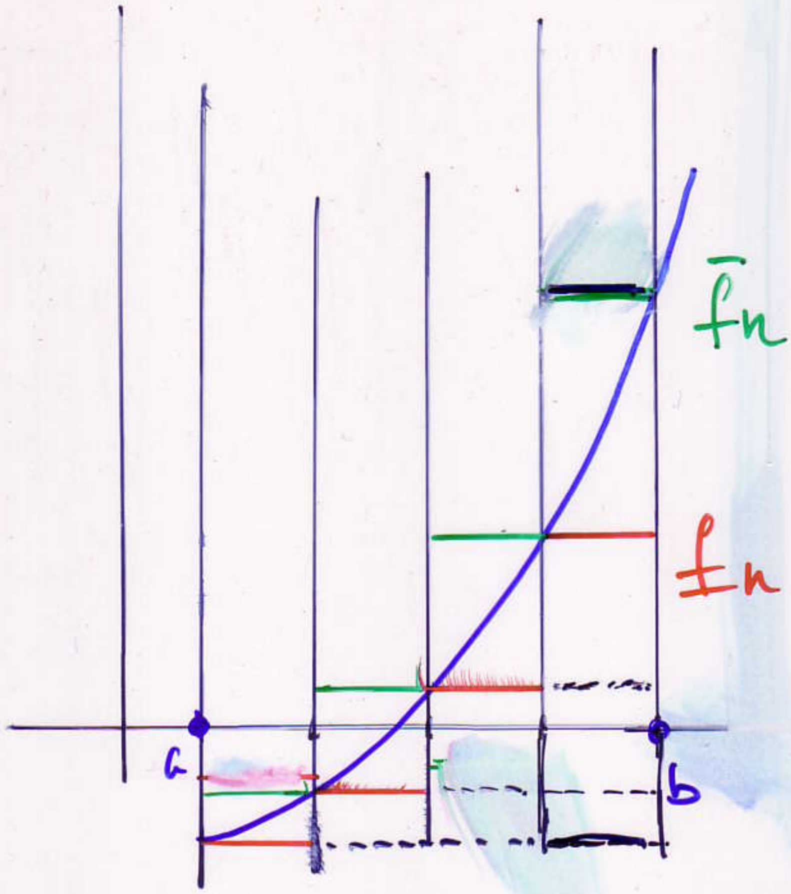
geg n_0: $\square + \square + \square + \square \le \varepsilon/3$

$\leadsto \delta$ mit $\square + \square + \square \le \varepsilon/3$

$$\int \underline{f}_{n_0} - \frac{\varepsilon}{3} \le \int T(\vec{\xi}) \le \int \overline{f}_{n_0} + \frac{\varepsilon}{3}$$

$$\le \quad c \quad \le$$

$$\Rightarrow |c - R(\vec{\xi})| \le \varepsilon$$



$$|\Delta x| = \frac{a-b}{n}$$

$$\square = \int \bar{f}_n - \int \underline{f}_n \leq |\Delta x| \left(f(b) - f(a) \right) \to 0$$

monoton $\Rightarrow$ integrierbar

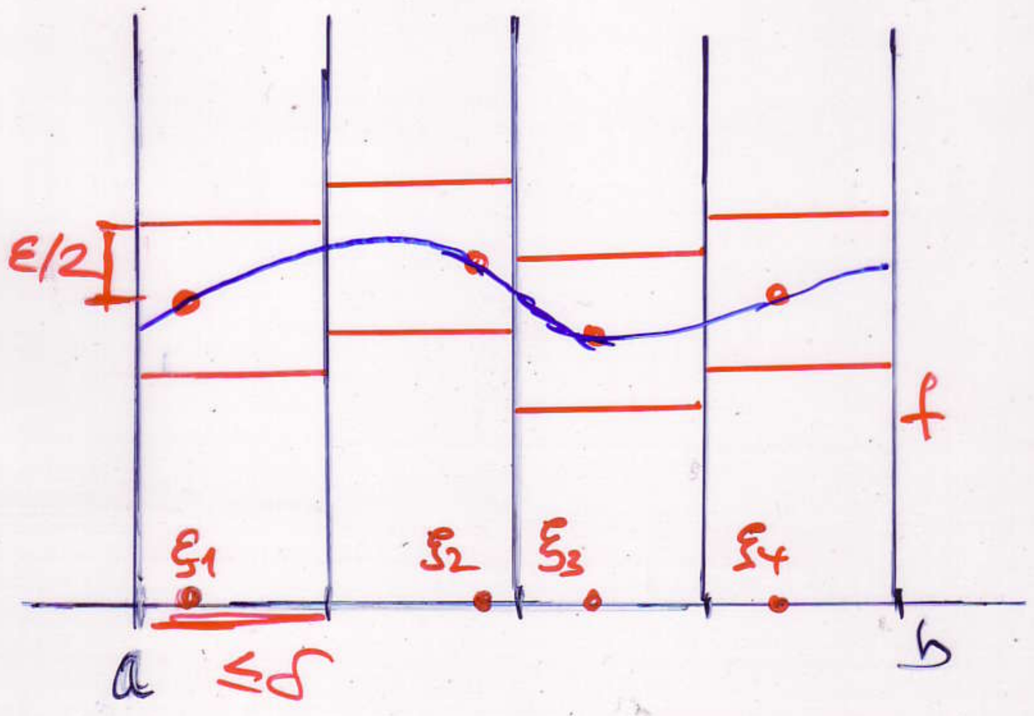
Satz 15.10 *Jede stetige Funktion $f : [a,b] \to \mathbb{R}$ ist integrierbar und es gilt*

(i) Zu jedem $\varepsilon > 0$ gibt es $\delta > 0$ so, dass für alle $a \leq c \leq d \leq b$ mit $d - c \leq \delta$ und alle $\xi \in [c,d]$ gilt

$$\left|\int_c^d f(x)\mathrm{d}x - f(\xi)(d-c)\right| \leq \varepsilon(d-c)$$

(ii) Mittelwertsatz*: Es gibt $\xi \in [a,b]$ mit $\int_a^n f(x)\mathrm{d}x = f(\xi)(b-a)$.*

Bew. $\varepsilon = \frac{1}{n}$ $\quad \delta$ zu $\varepsilon/2$



$\int \bar{f}_n - \int \underline{f}_n \leq \qquad \to 0$