Fachbereich Mathematik
AG Algebra, Geometrie, Funktionalanalysis
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SS 2009

TECHNISCHE UNIVERSITÄT DARMSTADT

18. Juni 2009

9. Problem sheet on "Lie Groups and Their Representations"

Exercise 9.1 Let X be an infinite set and $S_{(X)}$ be the group of all those permutations φ of X moving only finitely many points, i.e.,

$$|\{x \in X \colon \varphi(x) \neq x\}| < \infty.$$

Show that for each element $\varphi \neq \mathrm{id}_X$ in $S_{(X)}$ the conjugacy class

$$C_\varphi := \{\psi\varphi\psi^{-1} \colon \psi \in S_{(X)}\}$$

is infinite. Hint: Consider a description of φ in terms of cycles. In view of Example 5.3.16, this implies that the left/right regular representation of the discrete group $S_{(X)}$ is a factor representation not of type I.

Exercise 9.2 Let $\mathcal{H}$ be an infinite dimensional Hilbert space. Show that every trace functional $T \colon B(\mathcal{H}) \to \mathbb{C}$ vanishes in $\mathbf{1}$, i.e.,

$$T(AB) = T(BA) \quad \text{for} \quad A, B \in B(\mathcal{H})$$

implies $T(\mathbf{1}) = 0$. Here are some steps to follow:

(a) T is conjugation invariant, i.e., $T(gAg^{-1}) = T(A)$ for $g \in \mathrm{GL}(\mathcal{H})$ and $A \in B(\mathcal{H})$.

(b) If P and Q are two orthogonal projections in $B(\mathcal{H})$ for which theare are unitary isomorphisms $P(\mathcal{H}) \to Q(\mathcal{H})$ and $P(\mathcal{H})^\perp \to Q(\mathcal{H})^\perp$, then $T(P) = T(Q)$.

(c) For each $n \in \mathbb{N}$, there exists a unitary isomorphism $u_n \colon \mathcal{H} \to \mathcal{H}^n$, i.e.,

$$\mathcal{H} = \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_n \quad \text{with} \quad \mathcal{H}_j \cong \mathcal{H}.$$

Let $P_j^{(n)}$ denote the orthogonal projection onto $\mathcal{H}_j$.

(d) Show that $T(P_j^{(n)}) = \frac{1}{n}T(\mathbf{1})$ and use (b) to derive $T(P_1^{(2)}) = T(P_1^{(3)})$. Conclude that $T(\mathbf{1}) = 0$.

Exercise 9.3 Let G be a topological group, $K \subseteq G$ be a closed subgroup and $X := G/K$ the corresponding homogeneous space with base point $x_0 := \mathbf{1}K$. We fix a 1-cocycle $J \colon G \times X \to \mathbb{C}^\times$ and $0 \neq Q \in \mathcal{P}(X, \sigma, J)$, so that

$$(\pi(g)f)(x) := J(g, x)f(g^{-1}.x)$$

defines a unitary representation of G on $\mathcal{H}_Q \subseteq \mathbb{C}^X$ (Proposition 5.1.5). Show that:

(a) $\chi(k) := J(k, x_0)$ defines a character $\chi \colon K \to \mathbb{T}$.

(b) $\mathcal{H}_{Q,\chi} := \bigcap_{k\in K} \ker(\pi(k) - \chi(k)\mathbf{1}) \neq \{0\}$. It generates $\mathcal{H}_Q$ under the G-action.

(c) If $\mathcal{H}_{Q,\chi}$ is one dimensional, then the G-representation on $\mathcal{H}_Q$ is irreducible.

Hint: Proposition 5.3.10.

Exercise 9.4 Let G be a topological group, $O \subseteq G$ be an open subset and $S \subseteq G$ any subset. Then the subsets OS and SO of G are open. Hint: $OS = \bigcup_{s \in S} Os$.

Exercise 9.5 Let G be a topological group and $K \subseteq G$ be a closed subgroup. We endow G/K with the quotient topology, i.e., $O \subseteq G/K$ is open if and only if $q^{-1}(O) \subseteq G$ is open, where $q\colon G \to G/K$ is the quotient map. Show that:

(a) The quotient map $q\colon G \to G/K$ is open. Hint: Exercise 9.4.

(b) To see that G/K is Hausdorff, argue that for $y \not\in xK$ there exists an open $\mathbf{1}$-neighborhood U in G with $U^{-1}Uy \cap xK = \emptyset$ and derive that $\pi(Uy) \cap \pi(Ux) = \emptyset$.

(c) The action $\sigma\colon G \times G/K \to G/K, (g, xK) \mapsto gxK$ is continuous. Hint: (a) and the openness of $\mathrm{id}_G \times q$.

(d) The map $q \times q\colon G \times G \to G/K \times G/K$ is an open map, i.e., $O \subseteq G/K \times G/K$ is open if and only if $(q \times q)^{-1}(O)$ is open in $G \times G$.

(e) Show that for every continuous K-biinvariant function $\varphi\colon G \to \mathbb{C}$, the kernel $Q(xK, yK) := \varphi(xy^{-1})$ on $G/K \times G/K$ is continuous.

Exercise 9.6 Let $\sigma\colon G \times X \to X, (g, x) \mapsto g.x$ be a transitive continuous action of the topological group G on the topological space X. Fix $x_0 \in X$ and let

$$K := \{g \in G\colon g.x_0 = x_0\}$$

be the stabilizer subgroup of x_0. Show that:

(a) We obtain a continuous bijective map $\eta\colon G/K \to X, gK \mapsto g.x_0$.

(b) Suppose that η has a continuous local section, i.e., x_0 has a neighborhood U for which there exists a continuous map $\tau\colon U \to G$ with $\tau(y).x_0 = y$ for $y \in U$. Then η is open, hence a homeomorphism.

(c) Let $G := \mathbb{R}_d$ be the group $(\mathbb{R}, +)$, endowed with the discrete topology and $X := \mathbb{R}$, endowed with the canonical topology. Then $\sigma(x, y) := x + y$ defines a continuous transitive action of G on X for which the orbit map η is continuous and bijective but not open.

Exercise 9.7 Let V be a euclidean space, $\mathbb{S} \subseteq V$ be its unit sphere, $G := \mathrm{O}(V)$ be its orthogonal group, endowed with the strong operator topology, $e_0 \in \mathbb{S}$ and $K \cong \mathrm{O}(e_0^\perp)$ be the stabilizer of e_0 in G. Show that the orbit map $\sigma^{e_0}\colon \mathrm{O}(V) \to \mathbb{S}, g \mapsto ge_0$ induces a homeomorphism

$$\eta\colon G/K = \mathrm{O}(V)/\mathrm{O}(e_0^\perp) \to \mathbb{S}, \quad gK \mapsto ge_0.$$

Hint: Show first that for $U := \mathbb{S} \setminus \{-e_0\}$ the map

$$\sigma\colon U \to \mathrm{O}(V), \quad \sigma(z)(v) := 2\frac{\langle v, e_0 + z\rangle}{\|e_0 + z\|^2}(e_0 + z) - v$$

is continuous and satisfies

$$\sigma(z)(e_0) = z.$$

Then apply Exercise 9.7.