Fachbereich Mathematik
AG Algebra, Geometrie, Funktionalanalysis
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8. Problem sheet on "Lie Groups and Their Representations"

Exercise 8.1 Let $C \subseteq V$ be a convex cone in the real vector space V and $\alpha \in V^*$ with $\alpha(c) > 0$ for $0 \neq c \in C$. Show that

$$S := \{c \in C \colon \alpha(c) = 1\}$$

satisfies:

(a) $C = \mathbb{R}_+ S$.

(b) $x \in S$ is an extreme point of S if and only if $\mathbb{R}_+ x$ is an extremal ray of C.

Exercise 8.2 Let $\mathcal{D}_1, \mathcal{D}_2 \subseteq \mathbb{C}$ be two open subsets and $\varphi \colon \mathcal{D}_1 \to \mathcal{D}_2$ be a biholomorphic map, i.e., φ is bijective and φ^{-1} is also holomorphic. Let $\mathcal{B}(\mathcal{D}) := L^2(\mathcal{D}, dz) \cap \mathcal{O}(\mathcal{D})$ denote the *Bergman space* of $\mathcal{D}$. Show that the map

$$\Phi \colon \mathcal{B}(\mathcal{D}_2) \to \mathcal{B}(\mathcal{D}_1), \quad f \mapsto (\varphi^* f) \cdot \varphi', \quad \varphi^* f = f \circ \varphi$$

is unitary. Hint: For the real linear map $\lambda_z \colon \mathbb{C} \to \mathbb{C}, w \mapsto zw$, we have $\det_{\mathbb{R}}(\lambda_z) = |z|^2$.

Exercise 8.3 Let $G = N \rtimes_\alpha K$ be a semidirect product group and $\varphi \in \mathcal{P}(N)$ be a positive definite function on N which is K-invariant in the sense that

$$\varphi(k.n) = \varphi(n) \quad \text{for} \quad k \in K, n \in N.$$

Then

$$\psi \colon G \to \mathbb{C}, \quad \psi(n,k) := \varphi(n)$$

is a positive definite function on G. Hint: Show that the representation $(\pi_\varphi, \mathcal{H}_\varphi)$ of N extends by $\pi_\varphi(k) f := f \circ \alpha(k)^{-1}$ to a unitary representation $(\pi_\varphi, \mathcal{H}_\varphi)$ of G (Proposition 5.1.5) and consider $\pi_\varphi \in \mathcal{P}(G)$.

Exercise 8.4 Show that for a euclidean space V, the group $\mathrm{O}(V)$ of linear surjective isometries acts transitively on the sphere

$$\mathbb{S}(V) = \{v \in V \colon \|v\| = 1\}.$$

Hint: For a unit vector $v \in \mathbb{S}(V)$ consider the map

$$\sigma_v(x) := x - 2\langle x, v\rangle v.$$

Show that $\sigma_v \in \mathrm{O}(V)$ and that for $x, y \in \mathbb{S}(V)$ there exists a $v \in \mathbb{S}$ with $\sigma_v(x) = y$.

Exercise 8.5 We consider the group $G := \mathrm{GL}_2(\mathbb{C})$ and the *complex projective line* (the *Riemann sphere*)

$$\mathbb{P}_1(\mathbb{C}) = \{[v] := \mathbb{C}v \colon 0 \neq v \in \mathbb{C}^2\}$$

of 1-dimensional linear subspaces of $\mathbb{C}^2$. We write $[x : y]$ for the line $\mathbb{C}\begin{pmatrix} x \\ y \end{pmatrix}$. Show that:

(a) The map $\mathbb{C} \to \mathbb{P}_1(\mathbb{C}), z \mapsto [z : 1]$ is injective and its complement consists of the single point $\infty := [1 : 0]$ (the horizontal line). We thus identify $\mathbb{P}_1(\mathbb{C})$ with the one-point compactification $\widehat{\mathbb{C}}$ of $\mathbb{C}$. These are the so-called *homogeneous coordinates* on $\mathbb{P}_1(\mathbb{C})$.

(b) The natural action of $\mathrm{GL}_2(\mathbb{C})$ on $\mathbb{P}_1(\mathbb{C})$ by $g.[v] := [gv]$ is given in the coordinates of (b) by

$$g.z = \sigma_g(z) := \frac{az + b}{cz + d} \quad \text{for} \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

(c) On $\mathbb{C}^2$ we consider the indefinte hermitian form

$$\beta(z, w) := z_1\overline{w_1} - z_2\overline{w_2} = w^* \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} z.$$

We define

$$\mathrm{U}_{1,1}(\mathbb{C}) := \{g \in \mathrm{GL}_2(\mathbb{C}) \colon (\forall z, w \in \mathbb{C}^2)\, \beta(gz, gw) = \beta(z, w)\}.$$

Show that $g \in \mathrm{U}_{1,1}(\mathbb{C})$ is equivalent to

$$g^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} g^* \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Show further that the above relation is equivalent to

$$\det g \in \mathbb{T}, \quad d = \overline{a} \det g \quad \text{and} \quad c = \overline{b} \det g.$$

In particular, we obtain $|a|^2 - |b|^2 = 1$.

(d) The hermitian form β is negative definite on the subspace $[z_1 : z_2]$ if and only if $|z_1| < |z_2|$, i.e., $[z_1 : z_2] = [z : 1]$ for $|z| < 1$. Conclude that $g.z := \frac{az+b}{cz+d}$ defines an action of $\mathrm{U}_{1,1}(\mathbb{C})$ on the open unit disc $\mathcal{D}$ in $\mathbb{C}$.

(e) Show that the action of the subgroup $\mathrm{SL}_2(\mathbb{R})$ of $\mathrm{SL}_2(\mathbb{C})$ on $\widehat{\mathbb{C}}$ preserves the circle $\widehat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ and the upper half plane

$$\mathbb{C}_+ := \{z \in \mathbb{C} \colon \operatorname{Im} z > 0\}.$$