Fachbereich Mathematik
AG Algebra, Geometrie, Funktionalanalysis
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7. Problem sheet on "Lie Groups and Their Representations"

Exercise 6.1 We have defined the tensor product $\mathcal{H}\widehat{\otimes}\mathcal{K}$ of two Hilbert spaces as a space of functions on the product $\mathcal{H} \times \mathcal{K}$, defined by the kernel

$$K((x',y'),(x,y)) = \langle x, x'\rangle\langle y, y'\rangle.$$

Show that $\mathcal{H}\widehat{\otimes}\mathcal{K}$ consists of continuous maps which are biantilinear, i.e., antilinear in each argument.

Exercise 6.2 Let V be a real vector space and $\omega\colon V \times V \to \mathbb{R}$ be a bilinear map.

(a) Show that on $\mathbb{R} \times V$ we obtain a group structure by

$$(t,v)(s,w) := (t + s + \omega(v,w), v + w).$$

This group is called the *Heisenberg group* $\mathrm{Heis}(V,\omega)$.

More generally, we obtain for any two abelian groups V and Z and any biadditive map $\omega\colon V \times V \to Z$ a group structure on $Z \times V$ by

$$(t,v)(s,w) := (t + s + \omega(v,w), v + w).$$

(b) Let $\mathcal{H}$ be a complex Hilbert space. How do we have to choose V and ω to obtain an isomorphism $\mathrm{Heis}(V,\omega) \cong \mathrm{Heis}(\mathcal{H})$?

(c) Verify that $Z(\mathrm{Heis}(V,\omega)) = \mathbb{R} \times \mathrm{rad}(\omega_s)$, where

$$\omega_s(v,w) := \omega(v,w) - \omega(w,v) \quad \text{and} \quad \mathrm{rad}(\omega_s) := \{v \in V\colon \omega_s(v,V) = \{0\}\}.$$

(d) Show that for $V = \mathbb{R}^2$ with $\omega(x,y) = x_1y_2$, the Heisenberg group $H(V,\omega)$ is isomorphic to the matrix group

$$H := \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x,y,z \in \mathbb{R} \right\}.$$

Exercise 6.3 Let $K\colon X \times X \to \mathbb{C}$ be a positive definite kernel and $\theta\colon X \to \mathbb{C}^\times$ a function. Determine necessary and sufficient conditions on θ such that

$$\theta(x)K(x,y)\overline{\theta(y)} = K(x,y) \quad \text{for} \quad x,y \in X.$$

Hint: Consider the subset $X_1 := \{x \in X\colon K(x,x) > 0\}$ and its complement X_0 separately.

Exercise 6.4 Show that if $(\pi,\mathcal{H})$ is a factor representation of G and there exists an irreducible subrepresentation $\mathcal{H}_1 \subseteq \mathcal{H}$, then $(\pi,\mathcal{H})$ is of type I. Hint: Consider the decomposition $\mathcal{H} = \mathcal{H}_d \oplus \mathcal{H}_c$ into continuous and discrete part and show that $\mathcal{H}_c$ is trivial.

Exercise 6.5 Let $(V, \|\cdot\|)$ be a normed space,

$$\mathbb{P}(V) := \{[v] := \mathbb{R}v \colon 0 \neq v \in V\}$$

be the space of one-dimensional subspace of V (the projective space). Show that

(a) $g.[v] := [gv]$ defines an action of $\mathrm{GL}(V)$ on $\mathbb{P}(V)$.

(b) $J \colon \mathrm{GL}(V) \times \mathbb{P}(V) \to \mathbb{R}^\times, J(g, [v]) := \frac{\|gv\|}{\|v\|}$ is a 1-cocycle with respect to this action.

Exercise 6.6 Let $G = \prod_{j \in J} G_j$ be a product of abelian topological groups and $p_j \colon G \to G_j$ be the projection maps. Show that the map

$$S \colon \bigoplus_{j \in J} \widehat{G_j} \to \widehat{G}, \quad (\chi_j)_{j \in J} \mapsto \prod_{j \in J} (\chi_j \circ p_j)$$

is an isomorphism of abelian groups.

Exercise 6.7 On $\mathbb{R}^n$ we consider the vector space $\mathcal{P}_k$ of all homogeneous polynomials of degree k:

$$p(x) = \sum_{|\alpha|=k} c_\alpha x^\alpha, \quad c_\alpha \in \mathbb{R}, x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}, \quad |\alpha| = \alpha_1 + \cdots + \alpha_n.$$

We associate to such a polynomial p a differential operator by

$$p(\partial) := \sum_{|\alpha|=k} c_\alpha \partial^\alpha, \quad \partial^\alpha := \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}, \quad \partial_i := \frac{\partial}{\partial x_i}.$$

Show that the *Fischer inner product*

$$\langle p, q \rangle := (p(\partial)q)(0)$$

defines on $\mathcal{P}_k$ the structure of a real Hilbert space with continuous point evaluations. Show further that its kernel is given by

$$K(x, y) = \frac{1}{k!} \langle x, y \rangle^k = \frac{1}{k!} \Big(\sum_{j=1}^n x_j y_j \Big)^k.$$

Hint: Show that the monomials $p_\alpha(x) = x^\alpha$ form an orthogonal subset with $\langle p_\alpha, p_\alpha \rangle = \alpha!$ and conclude with Theorem 3.1.3 that $K(x, y) = \sum_{|\alpha|=m} \frac{x^\alpha y^\alpha}{\alpha!}$.

Exercise 6.8 Let $K, Q \in \mathcal{P}(X, \mathbb{C})$ be positive definite kernels on X and $\theta \colon X \to \mathbb{C}^\times$. Show that

$$m_\theta \colon \mathcal{H}_K \to \mathcal{H}_Q, \quad f \mapsto \theta f$$

defines a unitary map if and only if

$$Q(x, y) = \theta(x) K(x, y) \overline{\theta(y)} \quad \text{for} \quad x, y \in X.$$