Fachbereich Mathematik
AG Algebra, Geometrie, Funktionalanalysis
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6. Problem sheet on “Lie Groups and Their Representations”

Exercise 5.1 Let $(\mathcal{H}_j)_{j\in J}$ be a family of Hilbert spaces and $A_j \in B(\mathcal{H}_j)$. Suppose that $\sup_{j\in J} \|A_j\| < \infty$. Then $A(x_j) := (Ax_j)$ defines a bounded linear operator on $\widehat{\oplus}_{j\in J}\mathcal{H}_j$ with

$$\|A\| = \sup_{j\in J} \|A_j\|.$$

If, conversely, $\mathcal{H} = \widehat{\oplus}_{j\in J}\mathcal{H}_j$ is a Hilbert space direct sum and $A \in B(\mathcal{H})$ preserves each subspace $\mathcal{H}_j$, then the restrictions $A_j := A|_{\mathcal{H}_j}$ are bounded operators in $B(\mathcal{H}_j)$ satisfying $\|A\| = \sup_{j\in J} \|A_j\|$.

Exercise 5.2 Let $(\pi, \mathcal{H})$ be an irreducible representation of the involutive semigroup $(S, *)$ and $\pi_n := \oplus_{j=1}^n \pi$ be the n-fold direct sum of π with itself on $\mathcal{H}^n = \oplus_{j=1}^n \mathcal{H}$. Show that

$$\pi_n(S)' \cong M_n(\mathbb{C}).$$

Hint: Write operators on $\mathcal{H}^n$ as matrices with entries in $B(\mathcal{H})$ (cf. Exercise 1.3.6) and evaluate the commuting condition.

Exercise 5.3 Let V be a real topological vector space. Show that each continuous character $\chi\colon V \to \mathbb{T}$ is of the form $\chi(v) = e^{i\alpha(v)}$ for some continuous linear functional $\alpha \in V'$. Hint: Let $U \subseteq V$ be a circular 0-neighborhood (circular means that $\lambda U \subseteq U$ for $|\lambda| \leq 1$; such neighborhoods form a local basis in 0) with $\operatorname{Re}\chi(v) > 0$ for $v \in U + U$. Define a continuous (!) function

$$L\colon U \to]-\pi, \pi[\subseteq \mathbb{R} \quad \text{by} \quad e^{iL(u)} = \chi(u).$$

Observe that $L(x + y) = L(x) + L(y)$ for $x, y \in U$ and use this to see that

$$\alpha(x) := \lim_{n\to\infty} nL\Big(\frac{x}{n}\Big)$$

is an additive extension of L to V. Now it remains to observe that continuous additive maps $V \to \mathbb{R}$ are linear functionals (prove $\mathbb{Q}$-linearity first).

Exercise 5.4 Show that if B is a compact subset of a Banach space E, then its closed convex hull $K := \overline{\operatorname{conv}(B)}$ is also compact. Hint: Since we are dealing with metric spaces, it suffices to show precompactness, i.e., that for each $\varepsilon > 0$, there exists a finite subset $F \subseteq K$ with $K \subseteq B_\varepsilon(F) := F + B_\varepsilon(0)$. Since B is compact, there exists a finite subset $F_B \subseteq B$ with $B \subseteq B_\varepsilon(F_B)$. Then $\operatorname{conv}(B) \subseteq \operatorname{conv}(F_B) + B_\varepsilon(0)$, and since $\operatorname{conv}(F_B)$ is compact (why?), $\operatorname{conv}(F_B) \subseteq B_\varepsilon(F)$ for a finite subset $F \subseteq \operatorname{conv}(F_B)$. This leads to $\operatorname{conv}(B) \subseteq F + B_{2\varepsilon}(0)$, which implies implies $K \subseteq F + B_{\leq 2\varepsilon}(0)$.

Exercise 5.5 Show that for each $n \in \mathbb{N}$ the unitary group

$$\mathrm{U}_n(\mathbb{C}) = \{g \in \mathrm{GL}_n(\mathbb{C}) \colon \mathbf{1} = g^*g = gg^*\}$$

is compact.

Exercise 5.6 Let $\mathcal{H}$ be a complex Hilbert space and $G \subseteq \mathrm{U}(\mathcal{H})_s$ be a closed subgroup. Show that G is compact if and only if $\mathcal{H}$ can be written as an orthogonal direct sum $\mathcal{H} = \widehat{\bigoplus}_{j\in J}\mathcal{H}_j$ of finite dimensional G-invariant subspaces. Hint: Use Tychonov's Theorem and Exercise 5.5 to see that for any family of finite dimensional Hilbert spaces $(\mathcal{H}_j)_{j\in J}$, the topological group $\prod_{j\in J}\mathrm{U}(\mathcal{H}_j)_s$ is compact.

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Exercise 5.7 Let X be a locally compact space, μ a positive Radon measure on X, $\mathcal{H}$ a Hilbert space and $f \in C_c(X, \mathcal{H})$ be a compactly supported continuous function.

(a) Prove the existence of the $\mathcal{H}$-valued integral

$$I := \int_X f(x)\, d\mu(x),$$

i.e., the existence of an element $I \in \mathcal{H}$ with

$$\langle v, I\rangle = \int_X \langle v, f(x)\rangle\, d\mu(x) \quad \text{for} \quad v \in \mathcal{H}.$$

Hint: Verify that the right hand side of the above expression is defined and show that it defines a continuous linear functional on $\mathcal{H}$.

(b) Show that, if μ is a probability measure, then

$$I \in \overline{\mathrm{conv}(f(X))}.$$

Hint: Use the Hahn–Banach Separation Theorem.